INDIANA UNIVERSITY INDIANAPOLIS

Formerly IUPUI

(Indiana University–Purdue University Indianapolis)

REAL ANALYSIS AND MEASURE THEORY

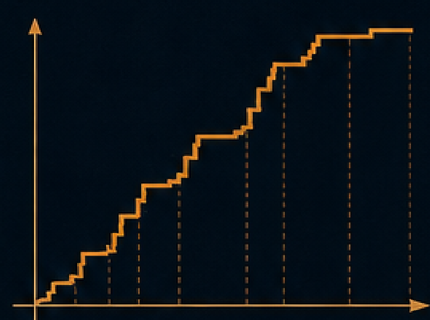
(MATH 54400)

SOLUTIONS TO THE QUALIFYING EXAMS

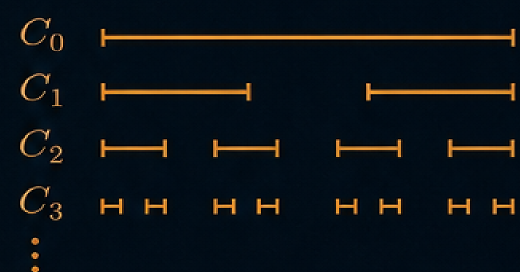
$$\int_X f\,d\mu = \lim_{n\to\infty}\int_X f_n\,d\mu$$

$$f^*(x) = \sup_{r>0}\frac{1}{2r}\int_{x-r}^{x+r}|f(y)|\,d\mu$$

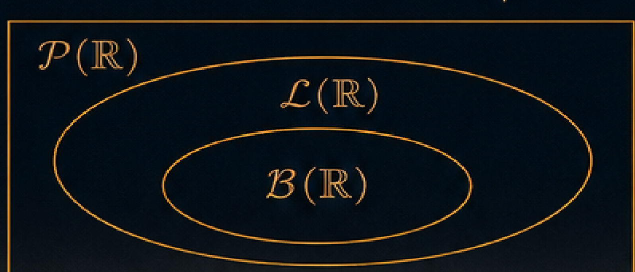
$$m^*(E) = \inf\left\{\sum_{k=1}^{\infty}|I_k| : E\subset\bigcup_{k=1}^{\infty} I_k,\ I_k \text{ intervals}\right\}$$



$$\frac{1}{x} = \int_0^\infty e^{-xt}\,dt \quad (x>0)$$



$$C = \bigcap_{n=0}^{\infty} C_n$$



REAL ANALYSIS

&

MEASURE THEORY

(MATH 54400/ MATH-I 544)

SOLUTIONS

Preface

This book is a collection of solutions to past qualifying examinations for Real Analysis and Measure Theory (MATH 54400 / MATH-I 544) at Indiana University Indianapolis (formerly IUPUI).

The solutions presented here are my own and were written independently as a way to study and practice graduate-level real analysis. They are not official solutions and do not necessarily reflect the methods or expectations of the faculty who prepared or graded the examinations.

Many of the problems have multiple valid approaches, and readers are encouraged to develop and compare their own proofs whenever possible. In many cases, shorter, more elegant proofs are possible.

Preparing these solutions has been an enjoyable way for me to revisit many beautiful results in measure theory, integration, functional analysis, and related areas. I hope that this collection will be useful to others studying these subjects.

Naturally, despite careful proofreading, errors or omissions may remain. I would appreciate hearing about any mistakes, typographical errors, or suggestions for improvement. If you find an error, typo, or a better solution, please contact me at dinepann at iu [dot] edu.

Last updated: August 2026

Diyath Pannipitiya

Contents

JANUARY 2026

1. Suppose that S is the smallest σ-algebra on $\mathbb{R}$ containing

$$\{[r, \infty) : r \in \mathbb{Q}\}.$$

Prove that S is the collection of Borel subsets of $\mathbb{R}$.

Proof. Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra. Let $\mathcal{B}(S)$ be the smallest σ-algebra on $\mathbb{R}$ containing $\{[r, \infty) : r \in \mathbb{Q}\}$.

Let $a \in \mathbb{R}$. Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of rational numbers such that

$$a_1 > a_2 > \cdots > a \qquad \text{and} \qquad a_n \to a.$$

Because the set of rational numbers is dense in $\mathbb{R}$, this is possible.

By definition, $[a_n, \infty) \in \mathcal{B}(S)$. Because $\mathcal{B}(S)$ is a σ-algebra, which is closed under taking countable unions, and

$$(a, \infty) = \bigcup_{n=1}^{\infty} [a_n, \infty),$$

we have

$$(a, \infty) \in \mathcal{B}(S).$$

Since $a \in \mathbb{R}$ is arbitrary and $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing all of the intervals of the form (a, ∞), we have

$$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{B}(S).$$

The converse inclusion

$$\mathcal{B}(S) \subseteq \mathcal{B}(\mathbb{R})$$

follows immediately from the fact

$$[r, \infty) = \bigcap_{n=1}^{\infty} \left(r - \frac{1}{n}, \infty \right)$$

for any $r \in \mathbb{Q}$. Thus,

$$\mathcal{B}(S) = \mathcal{B}(\mathbb{R}).$$

$\square$

2. Let (X, S, μ) be a measure space with $\mu(X) < \infty$, let $f : X \to [0, \infty)$ be an S-measurable function, and for each $i \in \mathbb{Z}_+$ let

$$m_i = \mu\left(\left\{x \in X : 2^{i-1} < f(x) \leq 2^i\right\}\right).$$

Show that

$$\int f \, d\mu < \infty$$

if and only if

$$\sum_{i=1}^{\infty} 2^i m_i < \infty.$$

Proof. For $i \in \mathbb{Z}_+$, define

$$X_i := \left\{x \in X : 2^{i-1} < f(x) \leq 2^i\right\}.$$

Then,

$$\bigcup_{i \in \mathbb{Z}_+} X_i \subseteq X.$$

($\Rightarrow$). Suppose $\int f \, d\mu < \infty$. Then,

$$\begin{aligned}
\sum_{i=1}^{\infty} 2^i m_i &= \sum_{i=1}^{\infty} \int_{X_i} 2^i d\mu \\
&= 2 \sum_{i=1}^{\infty} \int_{X_i} 2^{i-1} d\mu \\
&\leq 2 \sum_{i=1}^{\infty} \int_{X_i} f \, d\mu \\
&\leq 2 \int_X f \, d\mu \\
&< \infty.
\end{aligned}$$

($\Leftarrow$). Suppose $\sum_{i=1}^{\infty} 2^i m_i < \infty$. Because $\mu(X) < \infty$, we have

$$\mu(X_0) := \mu\left(\{x \in X : 0 \le f(x) \le 1\}\right) < \infty.$$

Thus,

$$\begin{aligned}
\int_X f \, d\mu &= \int_{X_0} f \, d\mu + \int_{\bigcup_{i \in \mathbb{Z}_+} X_i} f \, d\mu \\
&\le \int_{X_0} f \, d\mu + \int_{\bigcup_{i \in \mathbb{Z}_+} X_i} 2^i \, d\mu \\
&= \int_{X_0} f \, d\mu + \sum_{i=1}^{\infty} 2^i m_i \\
&< \infty.
\end{aligned}$$

Thus,

$$\int_X f \, d\mu < \infty \qquad \Longleftrightarrow \qquad \sum_{i=1}^{\infty} 2^i m_i < \infty.$$

$\square$

3. Compute the following limit and justify your answer:

$$\lim_{n \to \infty} \int_0^{\infty} \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) \, dx.$$

Proof 1.

Proof. By the change of variables $u = 1 + \frac{x}{n}$, we get

$$\int_0^{\infty} \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) \, dx = n \int_1^{\infty} \frac{\sin(u-1)}{u^n} \, du.$$

Notice that

$$\begin{aligned}
\left| n \int_1^{\infty} \frac{\sin(u-1)}{u^n} \, du \right| &= n \lim_{m \to \infty} \left| \int_1^{m} \frac{\sin(u-1)}{u^n} \, du \right| \\
&\le n \lim_{m \to \infty} \int_1^{m} \left| \frac{\sin(u-1)}{u^n} \right| \, du \\
&\le n \lim_{m \to \infty} \int_1^{m} \frac{u-1}{u^n} \, du
\end{aligned}$$

$$
\begin{aligned}
&= n \lim_{m\to\infty} \left[\frac{1}{(-n+2)u^{n-2}} - \frac{1}{(-n+1)u^{n-1}} \right]_1^m \\
&= n \left(\frac{1}{n-2} - \frac{1}{n-1} \right), \quad \text{for } n \geq 3 \\
&= \frac{n}{(n-2)(n-1)}.
\end{aligned}
$$

Thus,

$$0 \leq \lim_{n\to\infty} \left| \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\, dx \right| \leq \lim_{n\to\infty} \left(\frac{n}{(n-2)(n-1)} \right) = 0.$$

Therefore, by the Squeeze theorem,

$$\boxed{\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\, dx = 0.}$$

$\square$

Proof 2.

Proof. Define the sequences $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ of measurable functions by

$$f_n(x) := \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)$$

and

$$g_n(x) := \left(1+\frac{x}{n}\right)^{-n}$$

respectively.

Then we have

$$|f_n(x)| \leq g_n(x), \qquad \forall x \in \mathbb{R}, \forall n \in \mathbb{N}.$$

Because

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{n} = e^x \qquad \text{for any} \qquad x \in \mathbb{R},$$

we have

$$f(x) := \lim_{n\to\infty} f_n(x) = e^{-x} \cdot 0 = 0, \qquad \forall x \in \mathbb{R}$$

and

$$g(x) := \lim_{n\to\infty} g_n(x) = e^{-x}, \qquad \forall x \in \mathbb{R}.$$

Notice that,

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty g_n(x)\ dx &= \lim_{n\to\infty} \left(\lim_{m\to\infty} \int_0^m \left(1+\frac{x}{n}\right)^{-n}\ dx \right) \\
&= \lim_{n\to\infty} \left(\lim_{m\to\infty} \left[\frac{\left(1+\frac{x}{n}\right)^{-n+1}}{-n+1} \cdot n \right]_0^m \right) \\
&= \lim_{n\to\infty} \left(\frac{n}{-n+1} \lim_{m\to\infty} \left[\left(1+\frac{m}{n}\right)^{-n+1} - 1 \right] \right) \\
&= \lim_{n\to\infty} \left(\frac{n}{-n+1} (0-1) \right), \qquad \text{for all} \qquad n \geq 2 \\
&= \lim_{n\to\infty} \left(\frac{n}{n-1} \right) \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty g(x)\ dx &= \lim_{m\to\infty} \int_0^m e^{-x}\ dx \\
&= \lim_{m\to\infty} \left[-e^{-x} \right]_0^m \\
&= \lim_{m\to\infty} (-e^{-m}+1) \\
&= 1.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \int_0^\infty g_n(x)\ dx = \int_0^\infty g(x)\ dx. \tag{1}$$

$\square$

Therefore, by the general Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\ dx = \int_0^\infty f(x)\ dx = \int_0^\infty 0\ dx = 0.$$

So,

$$\boxed{\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\ dx = 0.}$$

4. Let (X, S, μ) be a measure space and let $E_1, E_2, \ldots$ be a sequence of measurable sets in X such that
$$\sum_{k=1}^{\infty} \mu(E_k) < \infty.$$
Show that μ almost every $x \in X$ lies in at most finitely many of the sets E_k.

Proof. Let
$$E := \{x \in X : x \in E_k \text{ for infinitely many } k \in \mathbb{N}\}.$$
Then,
$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$
Thus, $E \in S$. Notice that $\left(\bigcup_{k=n}^{\infty} E_k\right)_{n=1}^{\infty}$ is a decreasing sequence of measurable sets. Hence,
$$\begin{aligned}
\mu(E) &= \lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} E_k\right) \\
&\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) \\
&= 0, \qquad \because \sum_{k=1}^{\infty} \mu(E_k) < \infty.
\end{aligned}$$
Therefore, μ almost every $x \in X$ lies in at most finitely many of the sets E_k. □

5. Suppose $f \in L^1(\mathbb{R})$. Prove for Lebesgue a.e. $b \in \mathbb{R}$ that
$$|f(b)| \leq f^*(b),$$
where f^* denotes the Hardy–Littlewood maximal function associated to f.

Proof. **Recall.**

If $f \in L^1_{\text{loc}}(\mathbb{R})$, its Hardy–Littlewood maximal function is defined by
$$f^*(x) := \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(t)|\, dt.$$

Because $f \in L^1(\mathbb{R})$, we also have

$$|f| \in L^1(\mathbb{R}) \subseteq L^1_{\text{loc}}(\mathbb{R}).$$

Therefore, by the Lebesgue Differentiation Theorem, for Lebesgue almost every $b \in \mathbb{R}$,

$$|f(b)| = \lim_{r \to 0^+} \frac{1}{2r} \int_{b-r}^{b+r} |f(x)|\, dx.$$

On the other hand, by the definition of the Hardy–Littlewood maximal function,

$$f^*(b) = \sup_{r>0} \frac{1}{2r} \int_{b-r}^{b+r} |f(x)|\, dx.$$

Hence, for every $r > 0$,

$$\frac{1}{2r} \int_{b-r}^{b+r} |f(x)|\, dx \leq f^*(b).$$

Taking the limit as $r \to 0^+$ gives

$$|f(b)| \overset{\text{for } a.e}{=} \lim_{r \to 0^+} \frac{1}{2r} \int_{b-r}^{b+r} |f(x)|\, dx \leq f^*(b).$$

Therefore,

$$\boxed{|f(b)| \leq f^*(b)}$$

for Lebesgue almost every $b \in \mathbb{R}$.

□

6. Let $C^1([0,1])$ denote the normed space of continuously differentiable functions on $[0,1]$ with

$$\|f\| = \max_{x \in [0,1]} |f(x)|.$$

Prove that

$$\left(C^1([0,1]), \|\cdot\|\right)$$

is not a Banach space.

Proof. Define the sequence $(f_n)_{n=1}^{\infty}$ of functions $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) := \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ -n\left(x - \frac{1}{2}\right)^2 + 1 - \frac{1}{n}, & \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2} + \frac{1}{n} \\ -2x + 2, & \frac{1}{2} + \frac{1}{n} \leq x \leq 1. \end{cases}$$

Because

$$\lim_{x\to\left(\frac{1}{2}-\frac{1}{n}\right)^-} f_n(x) = \lim_{x\to\left(\frac{1}{2}-\frac{1}{n}\right)^-}(2x) = 1-\frac{2}{n},$$

and

$$\lim_{x\to\left(\frac{1}{2}-\frac{1}{n}\right)^+} f_n(x) = \lim_{x\to\left(\frac{1}{2}-\frac{1}{n}\right)^+}\left(-n\left(x-\frac{1}{2}\right)^2+1-\frac{1}{n}\right) = 1-\frac{2}{n},$$

we have

$$\lim_{x\to\left(\frac{1}{2}-\frac{1}{n}\right)} f_n(x) = 1-\frac{2}{n}.$$

Since

$$f_n\left(\frac{1}{2}-\frac{1}{n}\right) = 2\left(\frac{1}{2}-\frac{1}{n}\right) = 1-\frac{2}{n},$$

f_n is continuous at $x=\frac{1}{2}-\frac{1}{n}$.

Similarly, one can show that f_n is continuous at $x=\frac{1}{2}+\frac{1}{n}$.

Using the definition of the derivative one can show that

$$f_n'\left(\frac{1}{2}-\frac{1}{n}\right) = 2$$

and

$$f_n'\left(\frac{1}{2}+\frac{1}{n}\right) = -2$$

.

On $[0,1]\setminus\left\{\frac{1}{2}\pm\frac{1}{n}\right\}$, f_n is a polynomial.

Thus, f_n is differentiable on $[0,1]$. That is,

$$f_n \in C^1([0,1]).$$

Consider the tent map $f:[0,1]\to[0,1]$ defined by

$$f(x) := \begin{cases} 2x, & 0\le x\le \frac{1}{2} \\ -2x+2, & \frac{1}{2}< x\le 1. \end{cases}$$

Notice that

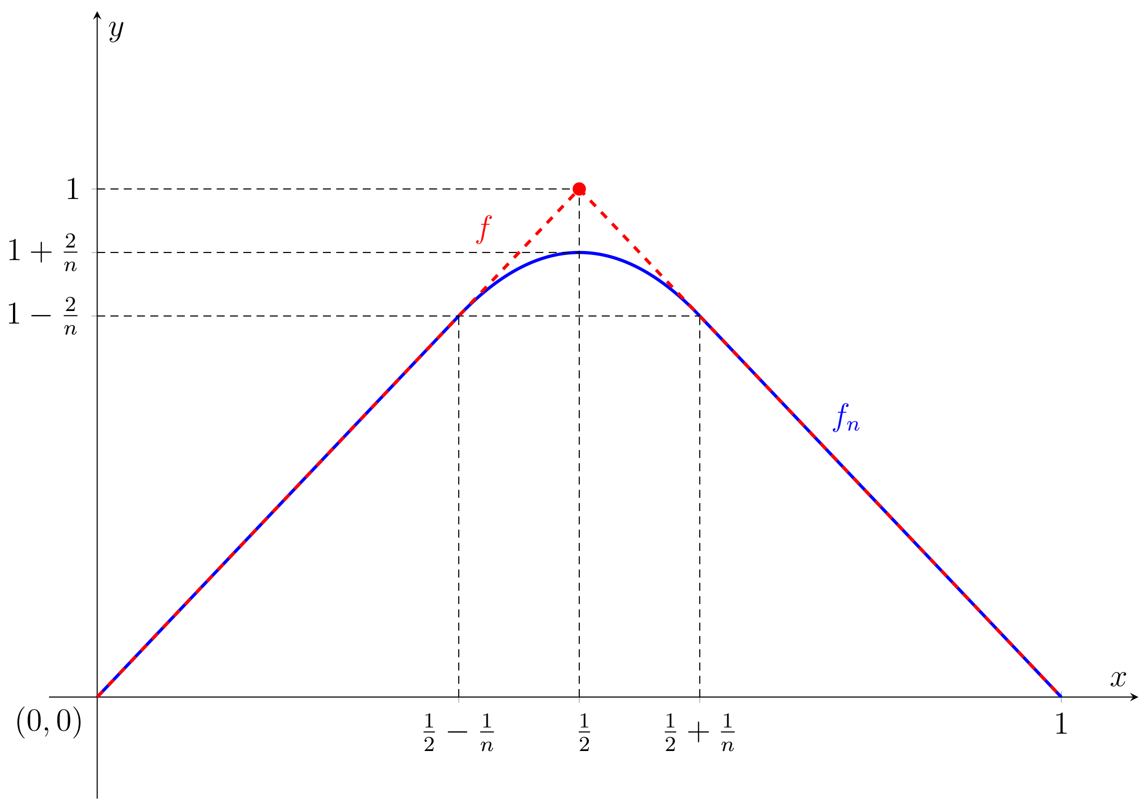
$$||f_n - f|| = 1-\left(1-\frac{1}{n}\right) = \frac{1}{n}.$$

Thus,

$$f_n \longrightarrow f \qquad \text{under} \qquad ||\cdot|| \quad \text{norm.}$$

However, $f \notin C^1([0,1])$ as f is not differentiable at $x = \dfrac{1}{2}$.

Therefore, $C^1([0,1])$ is not a Banach space under the given norm.



□

7. Compute

$$\lim_{n\to\infty} \int_0^n \frac{\sin(x)}{x}\, dx.$$

Hint. Use Fubini's Theorem and use the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt}\, dt$$

for any $x > 0$. (You can use that relation without proof.)

Answer.

Let $f_n(x,t) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$f_n(x,t) := \sin x e^{-xt}\chi_{[0,n]}(x).$$

First, we show that $f_n(x,t)$ is integrable over $\mathbb{R}^+ \times \mathbb{R}^+$ with respect to the product measure (2-dimensional Lebesgue measure):

$$\begin{aligned}
\int_{\mathbb{R}^+\times\mathbb{R}^+} |f_n(x)|\ dtdx &= \int_{\mathbb{R}^+\times\mathbb{R}^+} |\sin x e^{-xt}\chi_{[0,n]}(x)|\ dtdx \\
&\int_{\mathbb{R}^+} \left|\frac{\sin x}{x}\right| \chi_{[0,n]}(x)\ dx \\
&= \int_0^n \left|\frac{\sin x}{x}\right|\ dx
\end{aligned}$$

Because $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$, there is $h>0$ such that $\left|\dfrac{\sin x}{x} - 1\right| < \dfrac{1}{2}$ for all $x \in (0,h)$. Thus,

$$\begin{aligned}
\int_0^n \left|\frac{\sin x}{x}\right|\ dx &= \int_0^h \left|\frac{\sin x}{x}\right|\ dx + \int_h^n \left|\frac{\sin x}{x}\right|\ dx \\
&\le \int_0^h \frac{3}{2}\ dx + \int_h^n 1\ dx, \qquad \because |\sin x| \le |x| \\
&= \frac{3}{2}h + (n-h) \\
&= n + \frac{h}{2} \\
&< \infty.
\end{aligned}$$

Therefore, for each $n \in \mathbb{N}$, $f_n(x,t)$ is integrable over $\mathbb{R}^+ \times \mathbb{R}^+$. So, we can use the Fubini's theorem as follows.

$$\begin{aligned}
\int_{\mathbb{R}^+\times\mathbb{R}^+} f_n(x)\ dtdx &= \int_{\mathbb{R}^+\times\mathbb{R}^+} \sin x e^{-xt}\chi_{[0,n]}(x)\ dtdx \\
&= \int_0^n \int_0^\infty \sin x e^{-xt}\ dtdx \\
&= \int_0^\infty \int_0^n \sin x e^{-xt}\ dxdt, \qquad \text{by Fubini's theorem}
\end{aligned}$$

Using the Integration by Parts method twice we get

$$\int_0^n \sin x e^{-xt}\ dx = \left[-\frac{e^{-tx}\left(t\sin x + \cos x\right)}{1+t^2}\right]_0^n = \frac{1}{1+t^2}\left[1 - e^{-nt}(t\sin n + \cos n)\right]$$

Hence,

$$
\begin{aligned}
\int_{\mathbb{R}^+\times\mathbb{R}^+} f_n(x)\ dtdx &= \int_0^\infty \int_0^n \sin x e^{-xt}\ dxdt \\
&= \int_0^\infty \frac{1}{1+t^2}\left[1-e^{-nt}(t\sin n+\cos n)\right]\ dt \\
&= \int_0^\infty \left(\frac{1}{1+t^2} - e^{-nt}\cos n - e^{-nt}t\sin n\right)\ dt \\
&= \lim_{m\to\infty}\left[\tan^{-1} t + \frac{\cos n}{n}e^{-nt} - \frac{e^{-nt}\sin n}{n}(t+1)\right]_0^m \\
&= \frac{\pi}{2} - \frac{\cos n}{n} + \frac{\sin n}{n}.
\end{aligned}
$$

Thus,

$$
\lim_{n\to\infty}\int_0^n \frac{\sin(x)}{x}\,dx = \lim_{n\to\infty}\left(\frac{\pi}{2} - \frac{\cos n}{n} + \frac{\sin n}{n}\right) = \frac{\pi}{2}.
$$

$$
\boxed{\lim_{n\to\infty}\int_0^n \frac{\sin x}{x}\,dx = \frac{\pi}{2}.}
$$

AUGUST 2025

Throughout the exam, $\mathcal{L}$ denotes the Lebesgue σ-algebra on $\mathbb{R}$ and λ denotes the Lebesgue measure on $(\mathbb{R}, \mathcal{L})$.

1. Suppose that a, b, c, d are real numbers with $a < b$ and $c < d$ and let $|\cdot|$ denote the outer measure of a set. Using only the definition of outer measure, prove that

$$|(a, b) \cup (c, d)| = (b - a) + (d - c)$$

if and only if

$$(a, b) \cap (c, d) = \emptyset.$$

Proof. ($\Leftarrow$) Suppose

$$(a, b) \cap (c, d) = \emptyset.$$

From the definition of the outer-measure, we get

$$|(a, b)| = b - a \qquad \text{and} \qquad |(c, d)| = d - c.$$

Furthermore, we have

$$|(a, b) \cup (c, d)| := \inf \left\{ \sum_{k=1}^{\infty} |I_k| : I_k\text{'s are open intervals such that } (a, b) \cup (c, d) \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

Because (a, b) and (c, d) are also open intervals with

$$(a, b) \cup (c, d) \subseteq (a, b) \cup (c, d),$$

we get

$$|(a, b) \cup (c, d)| \leq |(a, b)| + |(c, d)| = (b - a) + (d - c). \tag{2}$$

Since (a,b) and (c,d) are disjoint, assume without loss of generality that $b \le c$, and choose $s \in [b,c]$. Let $\{O_k\}_{k=1}^{\infty}$ be any open-interval cover of $(a,b) \cup (c,d)$. Define

$$I_k = O_k \cap (-\infty, s) \qquad \text{and} \qquad J_k = O_k \cap (s, \infty),$$

discarding the empty sets. Then $\{I_k\}$ covers (a,b), $\{J_k\}$ covers (c,d), and the two unions are disjoint. Moreover,

$$|I_k| + |J_k| \le |O_k|$$

for every k. Therefore,

$$\sum_{k=1}^{\infty} |I_k| + \sum_{k=1}^{\infty} |J_k| \le \sum_{k=1}^{\infty} |O_k|.$$

Hence restricting the infimum to disjoint separated covers does not change its value.

Thus,

$$\begin{aligned} |(a,b) \cup (c,d)| := \inf \Bigg\{ & \sum_{k=1}^{\infty} (|I_k| + |J_k|) : I_k, J_k \text{ are open intervals,} \\ & (a,b) \subseteq \bigcup_{k=1}^{\infty} I_k, (c,d) \subseteq \bigcup_{k=1}^{\infty} J_k, \text{ and} \\ & \left(\bigcup_{k=1}^{\infty} I_k \right) \cap \left(\bigcup_{k=1}^{\infty} J_k \right) = \emptyset \Bigg\}. \end{aligned}$$

Hence,

$$\begin{aligned} |(a,b) \cup (c,d)| := \inf \Bigg\{ & \sum_{k=1}^{\infty} |I_k| + \sum_{k=1}^{\infty} |J_k| : I_k, J_k \text{ are open intervals,} \\ & (a,b) \subseteq \bigcup_{k=1}^{\infty} I_k, (c,d) \subseteq \bigcup_{k=1}^{\infty} J_k, \text{ and} \\ & \left(\bigcup_{k=1}^{\infty} I_k \right) \cap \left(\bigcup_{k=1}^{\infty} J_k \right) = \emptyset \Bigg\} \\ \ge \inf & \left\{ \sum_{k=1}^{\infty} |I_k| : I_k \text{ are open intervals such that } (a,b) \subseteq \bigcup_{k=1}^{\infty} I_k \right\} \\ + \inf & \left\{ \sum_{k=1}^{\infty} |J_k| : J_k \text{ are open intervals such that } (c,d) \subseteq \bigcup_{k=1}^{\infty} J_k \right\} \\ = {} & |(a,b)| + |(c,d)|. \end{aligned}$$

Thus,

$$|(a,b) \cup (c,d)| \geq (b-a) + (d-c). \tag{3}$$

Therefore by Equations (2) and (3) we get

$$|(a,b) \cup (c,d)| = (b-a) + (d-c).$$

($\Rightarrow$) Suppose

$$|(a,b) \cup (c,d)| = (b-a) + (d-c).$$

Assume $a < d$. It is clear that (a,b) cannot be inside (c,d) as it would then imply

$$(b-a) + (d-c) = |(a,b) \cup (c,d)| = |(c,d)| = (d-c),$$

which is a contradiction. Thus, if $(a,b) \cap (c,d) \neq \emptyset$, then we must have $a < c < b < d$ with $(a,b) \subset (a,d)$ and $(c,d) \subset (a,d)$. Then

$$(b-a) + (d-c) = |(a,b) \cup (c,d)| = |(a,d)| = (d-a).$$

This implies

$$b = c,$$

which is a contradiction as $c < b$. Therefore,

$$(a,b) \cap (c,d) = \emptyset.$$

This completes the proof. $\square$

2. Is there a measurable set $E \subset [0,1]$ such that for any $0 \leq a < b \leq 1$ we have

$$\lambda(E \cap [a,b]) = \frac{b-a}{2}\ ?$$

Either prove that such a set exists (an explicit example is preferred) or prove that such a set does not exist.

Proof. We prove such a set cannot exist. For the sake of contradiction, let's assume there exists a measurable set $E \subset [0,1]$ such that

$$\lambda(E \cap [a,b]) = \frac{b-a}{2}$$

for any non-degenerated (not singletons) closed interval $[a, b] \subseteq [0, 1]$.

It is clear that $\lambda(E) > 0$. Else, we get the contradiction

$$0 = \lambda(E) = \lambda(E \cap [0, 1]) = \frac{1-0}{2} = \frac{1}{2}.$$

We also have $\lambda(E) \neq 1$. If $\lambda(E) = 1$, then we get the contradiction

$$1 = \lambda(E) = \lambda(E \cap [0, 1]) = \frac{1-0}{2} = \frac{1}{2}.$$

For any $\epsilon > 0$ exists a collection $\{I_k\}_{k=1}^{\infty}$ of mutually disjoint open intervals of $[0, 1]$ such that

$$E \subset \bigcup_{k=1}^{\infty} I_k \qquad \text{and} \qquad \lambda(E) + 2\epsilon > \sum_{k=1}^{\infty} \lambda(I_k).$$

Using the hypothesis, then we get

$$\begin{aligned}
\lambda(E) &= \lambda\left(E \cap \bigcup_{k=1} I_k\right) \\
&= \lambda\left(\bigcup_{k=1} (E \cap I_k)\right) \\
&= \sum_{k=1}^{\infty} \lambda(E \cap I_k) \\
&= \sum_{k=1}^{\infty} \frac{\lambda(I_k)}{2} \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \lambda(I_k) \\
&< \frac{1}{2} \left(\lambda(E) + 2\epsilon\right) \\
&= \frac{\lambda(E)}{2} + \epsilon.
\end{aligned}$$

Thus,

$$\frac{\lambda(E)}{2} < \epsilon.$$

Because $\epsilon > 0$ is arbitrary, we get

$$\lambda(E) = 0.$$

This is a contradiction.

Therefore, such a set does not exist.

$\square$

3. Prove that there is no positive measurable function $f(x)$ on $[0,1]$ such that

$$\lim_{n\to\infty}\int [f(x)]^n \, d\lambda = 2.$$

Proof. Suppose, for contradiction, that such a function exists. Define the sequence $(f_n)_{n=1}^{\infty}$ of positive measurable functions by

$$f_n(x) := [f(x)]^n.$$

On

$$A := \{x \in [0,1] : f(x) > 1\},$$

f_n is increasing to ∞. Thus, we must have $\lambda(A) = 0$ because otherwise we get

$$\lim_{n\to\infty}\int [f(x)]^n \, d\lambda = \infty$$

by the Monotone Convergence Theorem.

On

$$B := \{x \in [0,1] : f(x) < 1\},$$

f_n is decreasing to 0. By using the Monotone Convergence Theorem (on the sequence $(f_1 - f_n)_{n=1}^{\infty}$) we see that

$$\int_B f_n \, d\lambda \to 0.$$

Thus, we must have

$$\begin{aligned}
2 &= \lim_{n\to\infty}\int_{[0,1]} f_n(x)\, dx \\
&= \lim_{n\to\infty}\int_A f_n(x)\, dx + \lim_{n\to\infty}\int_B f_n(x)\, dx + \lim_{n\to\infty}\int_{f^{-1}\{1\}} f_n(x)\, d\lambda \\
&= 0 + 0 + \lim_{n\to\infty}\int_{f^{-1}\{1\}} f_n(x)\, d\lambda \\
&= \lim_{n\to\infty}\int_{f^{-1}\{1\}} f_n(x)\, d\lambda.
\end{aligned}$$

We also have

$$\lim_{n\to\infty}\int_{f^{-1}\{1\}} f_n(x)\, d\lambda = \lim_{n\to\infty}\int_{f^{-1}\{1\}} 1^n \, d\lambda = \lambda\left(f^{-1}\{1\}\right).$$

Thus,

$$\lambda\left(f^{-1}\{1\}\right) = 2.$$

This is a contradiction as

$$\lambda\left(f^{-1}\{1\}\right) \leq \lambda\left([0,1]\right) = 1.$$

Therefore, such a function cannot exist. $\square$

4. Suppose that (X, S) is a measurable space and $f : X \to \mathbb{R}$ is a function. Let

$$\operatorname{graph}(f) \subset X \times \mathbb{R}$$

denote the graph of f:

$$\operatorname{graph}(f) = \{(x, f(x)) : x \in X\}.$$

Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of $\mathbb{R}$. Prove that

$$\operatorname{graph}(f) \in S \otimes \mathcal{B}$$

if f is an S-measurable function. (Here, $S \otimes \mathcal{B}$ denotes the product σ-algebra.)

Proof. Define the (coordinate) projections

$$\pi_X : X \times \mathbb{R} \to X, \qquad \pi_{\mathbb{R}} : X \times \mathbb{R} \to \mathbb{R}$$

by

$$\pi_X(x, y) = x \qquad \text{and} \qquad \pi_{\mathbb{R}}(x, y) = y$$

respectively.

Because

$$\pi_X^{-1}(E) = E \times \mathbb{R} \in \mathcal{S} \otimes \mathcal{B} \qquad \text{for every} \qquad E \in \mathcal{S},$$

the map π_X is $(\mathcal{S} \otimes \mathcal{B}, \mathcal{S})$-measurable.

Similarly, $\pi_{\mathbb{R}}$ is $(\mathcal{S} \otimes \mathcal{B}, \mathcal{B})$-measurable, since for every $B \in \mathcal{B}$, we have

$$\pi_{\mathbb{R}}^{-1}(B) = X \times B \in \mathcal{S} \otimes \mathcal{B}.$$

Because $f : X \to \mathbb{R}$ is $\mathcal{S}$-measurable, the composition

$$f \circ \pi_X : X \times \mathbb{R} \to \mathbb{R}$$

is $(\mathcal{S} \otimes \mathcal{B})$-measurable. Therefore, the function

$$F : X \times \mathbb{R} \to \mathbb{R}$$

defined by

$$F(x, y) = f(x) - y$$

is measurable, since

$$F = f \circ \pi_X - \pi_{\mathbb{R}}$$

is the difference of two measurable real-valued functions.

Notice that

$$\begin{aligned}
\operatorname{graph}(f) &= \{(x, y) \in X \times \mathbb{R} : y = f(x)\} \\
&= \{(x, y) \in X \times \mathbb{R} : f(x) - y = 0\} \\
&= \{(x, y) \in X \times \mathbb{R} : F(x, y) = 0\} \\
&= F^{-1}(\{0\}).
\end{aligned}$$

It is clear that $\{0\} \in \mathcal{B}$. Because F is measurable, we get

$$F^{-1}(\{0\}) \in \mathcal{S} \otimes \mathcal{B}.$$

That is:

$$\operatorname{graph}(f) \in \mathcal{S} \otimes \mathcal{B}.$$

□

5. Recall that ℓ^1 consists of all sequences $(a_1, a_2, a_3, \ldots)$ of real numbers such that

$$\sum_{k=1}^{\infty} |a_k| < \infty.$$

(a) Show that

$$\|(a_1, a_2, a_3, \ldots)\|_\infty = \sup_{k \in \mathbb{N}} |a_k|$$

defines a norm on ℓ^1.

(b) Prove that ℓ^1 with the norm $\|\cdot\|_\infty$ is not a Banach space.

Answer.

(a) **Positive Definiteness**

Let $\vec{a} \equiv (a_k)_{k=1}^{\infty} \in \ell^1$. Because we take the supremum of the absolute values of the terms a_k, we have $||\vec{a}||_\infty \geq 0$. Furthermore

$$\begin{aligned} ||\vec{a}||_\infty = 0 &\iff \sup_{k\in\mathbb{N}} |a_k| = 0 \\ &\iff |a_k| = 0 \quad \forall k \in \mathbb{N} \\ &\iff a_k = 0 \quad \forall k \in \mathbb{N} \\ \\ &\iff \vec{a} = 0. \end{aligned}$$

Homogeneity.
Let $c \in \mathbb{R}$. Then,

$$\begin{aligned} ||c\vec{a}||_\infty &= ||c(a_k)_{k=1}^\infty||_\infty \\ &= ||(ca_k)_{k=1}^\infty||_\infty \\ &= \sup_{k\in\mathbb{N}} |ca_k| \\ &= \sup_{k\in\mathbb{N}} |c| \cdot |a_k| \\ &= |c| \sup_{k\in\mathbb{N}} |a_k| \\ &= |c| ||\vec{a}||_\infty. \end{aligned}$$

Sub-additivity (Triangle Inequality).
Let $\vec{a} \equiv (a_k)_{k=1}^\infty$, $\vec{b} \equiv (b_k)_{k=1}^\infty \in \ell^1$. Then,

$$\begin{aligned} ||\vec{a} + \vec{b}||_\infty &= ||(a_k)_{k=1}^\infty + (b_k)_{k=1}^\infty||_\infty \\ &= ||(a_k + b_k)_{k=1}^\infty||_\infty \\ &= \sup_{k\in\mathbb{N}} |a_k + b_k| \\ &\leq \sup_{k\in\mathbb{N}} (|a_k| + |b_k|) \\ &\leq \sup_{k\in\mathbb{N}} |a_k| + \sup_{k\in\mathbb{N}} |b_k| \\ &= ||\vec{a}||_\infty + ||\vec{b}||_\infty. \end{aligned}$$

Thus, $||\cdot||_\infty$ defines a norm on ℓ^1.

(b) *Proof.* Define the sequence $(\vec{a}_n)_{n=1}^\infty$ of points in ℓ^1 by

$$\vec{a}_n \equiv ({}^n a_k)_{k=1}^\infty \equiv \left(1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n-1}, \frac{1}{n}, 0, 0, 0, \cdots\right).$$

That is, the first n elements of $\vec{a}_n$ are

$$1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n-1}, \frac{1}{n}$$

and the rest of the elements are 0.

Because

$$\sum_{k=1}^{\infty} |{}^n a_k| = \sum_{k=1}^{n} \frac{1}{k} + 0 = \sum_{k=1}^{n} \frac{1}{k} < \infty,$$

we have $\vec{a}_n \in \ell^1$ for all $n \in \mathbb{N}$.

For given $\epsilon > 0$, let $N \in \mathbb{N}$ be such that $\dfrac{1}{N} < \epsilon$. Let $m, n \in \mathbb{N}$ be such that $N < m < n$. Then,

$$\begin{aligned}
||\vec{a}_n - \vec{a}_m||_\infty &= \left|\left|\left(0, 0, 0, \cdots, \frac{1}{m+1}, \frac{1}{m+2}, \cdots, \frac{1}{n-1}, \frac{1}{n}, 0, 0, 0, \cdots\right)\right|\right|_\infty \\
&= \frac{1}{m+1} \\
&< \frac{1}{N} \\
&< \epsilon.
\end{aligned}$$

Thus, the sequence $(\vec{a}_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Let

$$\vec{a} \equiv \left(\frac{1}{k}\right)_{k=1}^{\infty}.$$

Then,

$$\begin{aligned}
||\vec{a}_n - \vec{a}||_\infty &= \left|\left|\left(0, 0, 0, \cdots, \frac{1}{n+1}, \frac{1}{n+2}, \frac{1}{n+3}, \cdots\right)\right|\right|_\infty \\
&= \frac{1}{n+1}.
\end{aligned}$$

Thus, $(\vec{a}_n)_{n=1}^{\infty}$ converges in $||\cdot||_\infty$ to $\vec{a}$. However, $\vec{a} \equiv \left(1, \frac{1}{2}, \frac{1}{3}, \cdots\right) \notin \ell^1$ because

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

Therefore, ℓ^1 is not a Banach space under $||\cdot||_\infty$ norm. $\square$

6. Recall that ℓ^∞ is the Banach space that consists of all bounded sequences

$$(a_1, a_2, a_3, \ldots)$$

of real numbers equipped with the norm

$$\|(a_1, a_2, a_3, \ldots)\|_\infty = \sup_{k\in\mathbb{N}} |a_k|.$$

Show that there exists a linear functional

$$\varphi : \ell^\infty \to \mathbb{R}$$

such that

$$|\varphi(a_1, a_2, a_3, \ldots)| \leq \|(a_1, a_2, a_3, \ldots)\|_\infty$$

for all $(a_1, a_2, a_3, \ldots) \in \ell^\infty$, and

$$\varphi(a_1, a_2, a_3, \ldots) = \lim_{k\to\infty} a_k$$

for all $(a_1, a_2, a_3, \ldots) \in \ell^\infty$ such that $\lim_{k\to\infty} a_k$ exists.

Proof. Let

$$A := \left\{(a_n) \in \ell^\infty : \lim_{n\to\infty} a_n \text{ exists}\right\}.$$

It is easy to verify that A is a linear subspace of ℓ^∞.

Define a linear functional

$$L : A \longrightarrow \mathbb{R}$$

by

$$L((a_n)) = \lim_{n\to\infty} a_n.$$

Since the limit of a sum is the sum of the limits and the limit of a scalar multiple is the corresponding scalar multiple, L is linear.

Next, suppose $(a_n) \in c$. Then

$$\begin{aligned}
|L((a_n))| &= \left|\lim_{n\to\infty} a_n\right| \\
&= \lim_{n\to\infty} |a_n| \\
&\leq \sup_{n\in\mathbb{N}} |a_n| \\
&= \|(a_n)\|_\infty.
\end{aligned}$$

Hence,

$$\|L\| \leq 1.$$

By the Hahn–Banach Theorem, then there exists a bounded linear functional

$$\varphi : \ell^\infty \longrightarrow \mathbb{R}$$

such that

$$\varphi|_A = L \qquad \text{and} \qquad \|\varphi\| = \|L\| \leq 1.$$

Therefore, for every $(a_n) \in \ell^\infty$, we have

$$|\varphi((a_n))| \leq \|\varphi\| \, \|(a_n)\|_\infty \leq \|(a_n)\|_\infty.$$

Finally, if $(a_n) \in A \subset \ell^\infty$, then

$$\varphi((a_n)) = L((a_n)) = \lim_{n\to\infty} a_n.$$

This completes the proof. $\square$

JANUARY 2025

Throughout the exam, λ denotes the Lebesgue measure on $(\mathbb{R}, \mathcal{L})$.

Problem 1. Show that the set of all real numbers that have a decimal expansion with the digit 5 appearing infinitely often is a Borel set.

Proof. Because the collection of all Borel sets is a σ-algebra, it is closed under taking complements. We will show that the set of all real numbers that have a decimal expansion with digit 5 appearing finitely many times is a Borel set.

Furthermore, since a σ-algebra is closed under taking countable unions, $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n, n+1]$, and $[n, n+1] = n + [0,1]$ for any $n \in \mathbb{Z}$, we will show that the set of all numbers in the interval $[0,1]$ that have a decimal expansion with digit 5 appearing finitely many times is a Borel set. For $k \in \mathbb{N}_0$, define

$$B_k := \{x \in [0,1] : \text{decimal expansion of } x \text{ has exactly } k \text{ number of 5's}\}.$$

Then,

$$\begin{aligned}
B_1 &= \frac{5 + B_0}{10} \cup \bigcup_{\substack{a_1=0 \\ a_1 \neq 5}}^{9} \frac{a_1 5 + B_0}{10^2} \cup \bigcup_{\substack{a_1,a_2=0 \\ a_1,a_2 \neq 5}}^{9} \frac{a_1 a_2 5 + B_0}{10^3} \cup \bigcup_{\substack{a_1,a_2,a_3=0 \\ a_1,a_2,a_3 \neq 5}}^{9} \frac{a_1 a_2 a_3 5 + B_0}{10^4} \cup \cdots \\
B_2 &= \frac{5 + B_1}{10} \cup \bigcup_{\substack{a_1=0 \\ a_1 \neq 5}}^{9} \frac{a_1 5 + B_1}{10^2} \cup \bigcup_{\substack{a_1,a_2=0 \\ a_1,a_2 \neq 5}}^{9} \frac{a_1 a_2 5 + B_1}{10^3} \cup \bigcup_{\substack{a_1,a_2,a_3=0 \\ a_1,a_2,a_3 \neq 5}}^{9} \frac{a_1 a_2 a_3 5 + B_1}{10^4} \cup \cdots \\
B_3 &= \frac{5 + B_2}{10} \cup \bigcup_{\substack{a_1=0 \\ a_1 \neq 5}}^{9} \frac{a_1 5 + B_2}{10^2} \cup \bigcup_{\substack{a_1,a_2=0 \\ a_1,a_2 \neq 5}}^{9} \frac{a_1 a_2 5 + B_2}{10^3} \cup \bigcup_{\substack{a_1,a_2,a_3=0 \\ a_1,a_2,a_3 \neq 5}}^{9} \frac{a_1 a_2 a_3 5 + B_2}{10^4} \cup \cdots \\
&\;\;\vdots
\end{aligned}$$

Here by a_1a_2, for example, we mean the two digits a_1 and a_2 of the number a_1a_2; not the multiplication $a_1 \times a_2$.

Thus, it suffices to show that B_0 is Borel. Let

$$
\begin{aligned}
I_0 &= [0,1] \\
I_1 &= I_0 \backslash [0.5, 0.6) = \bigcup_{\substack{m=0 \\ m\neq 5}}^{9} \frac{m + I_0}{10} \\
I_2 &= \bigcup_{\substack{m=0 \\ m\neq 5}}^{9} \frac{m + I_1}{10} \\
&\ \ . \\
&\ \ . \\
&\ \ . \\
I_{n+1} &= \bigcup_{\substack{m=0 \\ m\neq 5}}^{9} \frac{m + I_n}{10} \\
&\ \ . \\
&\ \ . \\
&\ \ .
\end{aligned}
$$

Then, $B_0 = \bigcap_{n=0}^{\infty} I_n$. Thus, B_0 is Borel. This completes the proof. $\square$

Problem 2. Suppose $(X, \mathcal{S}, \mu)$ is a measure space and $h \in \mathcal{L}^1(\mu)$. Prove for any $c > 0$ that

$$\mu\left(\{x \in X : |h(x)| \geq c\}\right) \leq \frac{1}{c}||h||_1.$$

Proof. Let $c > 0$. Then, $X = A \sqcup B$ where

$$A := \{x \in X : |h(x)| < c\} \ \text{ and } \ B := \{x \in X : |h(x)| \geq c\}.$$

Thus,

$$||h||_1 = ||h_{|A}||_1 + ||h_{|B}||_1 \geq ||h_{|B}||_1 = \int_B |h| \; d\mu \geq \int_B c \; d\mu = c \cdot \mu(B).$$

Therefore, $\mu(B) \leq \dfrac{1}{c}||h||_1$. $\square$

Problem 3. Prove the existence of the limit

$$\lim_{n\to\infty}\int_{\mathbb{R}}\frac{1}{1+x^2}\cdot\frac{e^{nx}}{1+e^{nx}}\,d\lambda(x)$$

and compute it.

Proof. For $n\in\mathbb{N}$, define $f_n:\mathbb{R}\to\mathbb{R}$ by $f_n(x):=\dfrac{1}{1+x^2}\cdot\dfrac{e^{nx}}{1+e^{nx}}$. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x):=\dfrac{1}{1+x^2}$. Then, $f_n\nearrow f$. Furthermore, we have the improper integral $\displaystyle\int_{\mathbb{R}}\frac{1}{1+x^2}\,d\lambda(x)=\pi$. Therefore, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_{\mathbb{R}}\frac{1}{1+x^2}\cdot\frac{e^{nx}}{1+e^{nx}}\,d\lambda(x)=\pi.$$

□

Problem 4. A sequence of measurable functions $f_n:[0,1]\to[-\infty,\infty]$ is said to **converge in measure** to a measurable function $f:[0,1]\to[-\infty,\infty]$ if

$$\text{for any } \delta>0,\ \lim_{n\to\infty}\lambda\{x\in[0,1]:|(f_n-f)(x)|>\delta\}=0.$$

Suppose that for each $n\in\mathbb{Z}^+$ we have $f_n\in\mathcal{L}^1([0,1])$ and also that $f\in\mathcal{L}^1([0,1])$. If $f_n\to f$ in measure, does it imply that

$$\lim_{n\to\infty}\int_{[0,1]}f_n\,d\lambda=\int_{[0,1]}f\,d\lambda?$$

Either prove it or give an explicit counterexample.

Answer. Let $f_n:=n\chi_{[0,\frac{1}{n}]}$ and $f\equiv 0$. Notice that

$\{x\in[0,1]:|(f_n-f)(x)|>\delta\}=\left\{x\in[0,1]:n\chi_{[0,\frac{1}{n}]}>\delta\right\}\subseteq\left[0,\dfrac{1}{n}\right]$ for large enough n. Thus,

$\mu\left(\{x\in[0,1]:|(f_n-f)(x)|>\delta\}\right)=\dfrac{1}{n}\to 0.$

This implies, $f_n\xrightarrow{\mu}f\equiv 0$.

But

$$\int_{[0,1]}f_n\,d\mu=n\int_{[0,1]}\chi_{[0,\frac{1}{n}]}\,d\mu=n\cdot\frac{1}{n}=1\nrightarrow 0=\int_{[0,1]}f\,d\mu.$$

Problem 5. Let $C([0,1])$ denote the vector space of all continuous functions $f:[0,1]\to\mathbb{R}$ with the operations of addition and scalar multiplication done pointwise. Prove that $C([0,1])$ with the norm defined by $\int_0^1 |f|$ is not a Banach space.

Proof. Define the sequence $(f_n)_{n=1}^{\infty}$ of functions by $f_n(x) := nx\chi_{[0,\frac{1}{n}]}(x) + \chi_{[\frac{1}{n},1]}(x)$. That is:

$$f_n(x) = \begin{cases} nx, & \text{if } 0 \le x \le \frac{1}{n}, \\ 1, & \text{if } \frac{1}{n} < x \le 1. \end{cases}$$

Also, define the function $f:[0,1]\to[0,1]$ by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } 0 < x \le 1. \end{cases}$$

Then,

$$\int_0^1 f_n \; d\mu = \int_0^{\frac{1}{n}} nx \; d\mu + \int_{\frac{1}{n}}^1 1 \; d\mu = \frac{1}{2n} + 1 - \frac{1}{n} \to 1 = \int_0^1 f \; d\mu.$$

□

But $f \notin C([0,1])$ even though $f_n \in C([0,1])$ for all $n \in \mathbb{N}$.

Problem 6. Let $\mathcal{L}^1(\mathbb{R})$ denote the Lebesgue measurable functions $f:\mathbb{R}\to\mathbb{R}$ such that $||f||_1 := \int |f| \; d\lambda < \infty$. Ignoring that there exist non-zero functions $f \in \mathcal{L}^1(\mathbb{R})$ with $||f||_1 = 0$, let us consider $\left(\mathcal{L}^1(\mathbb{R}), ||\cdot||_1\right)$ as a normed linear space.

(a). Show for Lebesgue almost every $x \in \mathbb{R}$ that

$$\int f(x-y)e^{-y^2} \; d\lambda(y)$$

is a finite number.

(b). Define a new function $Af:\mathbb{R}\to[-\infty,\infty]$ by $Af(x) = \int f(x-y)e^{-y^2} \; d\lambda(y)$. Prove that $Af \in \mathcal{L}^1(\mathbb{R})$.

(c). Show that A defines a bounded linear transformation $A:\mathcal{L}^1(\mathbb{R})\to\mathcal{L}^1(\mathbb{R})$ and show that $||A|| \le \sqrt{\pi}$.

Proof. (a). By change of variables we get

$$\begin{aligned}\left|\int f(x-y)e^{-y^2}\ d\lambda(y)\right| &= \left|\int f(y)e^{-(x-y)^2}\ d\mu(y)\right| \\ &\leq \int |f(y)\ d\mu(y)|\,,\ \because\ |e^{-(x-y)^2}| \leq 1 \\ &< \infty.\end{aligned}$$

Thus, the given integral is finite for almost every $x \in \mathbb{R}$.
Note. The *almost every* part comes from the fact:
"$f \in \mathcal{L}^1(\mathbb{R}) \Rightarrow f(x) < \infty$ for almost every $x \in \mathbb{R}$".

(b).

$$\begin{aligned}\int |Af|\ d\lambda(x) &= \int \left|\int f(x-y)e^{-y^2}\ d\lambda(y)\right|\ d\lambda(x) \\ &< \int\int \left|f(x-y)e^{-y^2}\right|\ d\lambda(y)\ d\lambda(x) \\ &= \int\int |f(x-y)|\cdot e^{-y^2}\ d\lambda(y)\ d\lambda(x) \\ &= \int\int |f(x-y)|\cdot e^{-y^2}\ d\lambda(x)\ d\lambda(y),\ \text{by part } (a) \text{ and Tonelli's Theorem} \\ &= \int e^{-y^2}\left[\int |f(x-y)|\ d\lambda(x)\right]\ d\lambda(y) \\ &= \int e^{-y^2}||f||_1\ d\lambda(y) \\ &= ||f||_1\int e^{-y^2}\ d\lambda(y) \\ &= ||f||_1\sqrt{\pi}.\end{aligned}$$

Note. To see why $\int e^{-y^2}\ d\lambda(y) = \sqrt{\pi}$.

Let $I := \int e^{-y^2}\ d\lambda(y)$. Then

$$\begin{aligned}I^2 &= \left(\int e^{-y^2}\ d\lambda(y)\right)\times\left(\int e^{-x^2}\ d\lambda(x)\right) \\ &= \int\int e^{-(x^2+y^2)}\ d\lambda(y)\ d\lambda(x) \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}r\ dr\ d\theta,\ \text{where } r^2 = x^2+y^2, \tan\theta = \frac{y}{x}. \\ &= \int_0^{2\pi}\left[\frac{e^{-r^2}}{2}\right]_{\infty}^{0}\ d\theta = \frac{1}{2}\int_0^{2\pi} d\theta \\ &= \pi.\end{aligned}$$

Thus, $I = \sqrt{\pi}$.

(c). For $f, g \in \mathcal{L}^1(\mathbb{R})$ and $K \in \mathbb{R}$ we have

$$\begin{aligned} A(f + Kg) &= \int [f(x-y) + Kg(x-y)]\, e^{-y^2}\ d\lambda(y) \\ &= \int f(x-y)e^{-y^2}\ d\lambda(y) + K \int g(x-y)e^{-y^2}\ d\lambda(y),\ \because \text{ integration is linear} \end{aligned}$$

Thus, A is linear. From part (b) we have $||Af|| \leq \sqrt{\pi}||f||_1$ for any $f \in \mathcal{L}^1(\mathbb{R})$. Thus, $||A|| \leq \sqrt{\pi}$.

$\square$

AUGUST 2024

Below, $m^*(\cdot)$ and $m(\cdot)$ are the outer Lebesgue and Lebesgue measures on the real line.

1. Let A be a non-measurable set. Show that there exists an open set O such that

$$A \subset O$$

and

$$m^*(O) < m^*(A) + m^*(O \setminus A).$$

Proof. For the sake of contradiction, suppose that for any open set $O \subseteq \mathbb{R}$ with $A \subset O$, we have

$$m(O) = m^*(A) + m^*(O \setminus A).$$

Then,

$$\inf_{\text{open } O \supset A} m(O) = \inf_{\text{open } O \supset A} \{m^*(A) + m^*(O \setminus A)\}.$$

From the definition of the outer measure of A, we have

$$\inf_{\text{open } O \supset A} m(O) = m^*(A).$$

Thus,

$$\begin{aligned} m^*(A) &= \inf_{\text{open } O \supset A} \{m^*(A) + m^*(O \setminus A)\} \\ &= m^*(A) + \inf_{\text{open } O \supset A} m^*(O \setminus A). \end{aligned}$$

Hence,

$$\inf_{\text{open } O \supset A} m^*(O \setminus A) = 0.$$

Therefore, we can find a decreasing sequence $(O_n)_{n=1}^{\infty}$ of open sets such that

$$O_n \supset A \qquad \text{and} \qquad m^*(O_n \setminus A) < \frac{1}{n}.$$

Let G be the G_δ set:

$$G := \bigcap_{n=1}^{\infty} O_n.$$

Because $G \setminus A = \bigcap_{n=1}^{\infty} (O_n \setminus A)$, we get

$$m^*(G \setminus A) < \frac{1}{n}.$$

Because this is true for any $n \in \mathbb{N}$,

$$m^*(G \setminus A) = 0.$$

Because Lebesgue measure is a complete measure, this implies $G \setminus A$ is measurable. Since

$$A = G \setminus (G \setminus A),$$

then A is also measurable. This is a contradiction. Thus, there is an open set $O \supset A$ such that

$$m(O) < m^*(A) + m^*(O \setminus A).$$

□

2. (a) Does f is continuous a.e. on $[0, 1]$ imply that

$$f = g \quad \text{a.e. on } [0, 1]$$

for some continuous function g?

(b) Does

$$f = g \quad \text{a.e. on } [0, 1]$$

for some continuous function g imply that f is continuous a.e. on $[0, 1]$?

Solution.

(a) We disprove the statement by giving a counterexample.

Consider the function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2}, \\ 1, & \frac{1}{2} \le x \le 1. \end{cases}$$

The function f is continuous at every point of $[0, 1]$ except $x = \frac{1}{2}$. Hence f is continuous almost everywhere on $[0, 1]$. Suppose, for a sake of contradiction, that there exists a continuous function $g : [0, 1] \to \mathbb{R}$ such that

$$f \overset{a.e}{=} g \qquad \text{on } [0, 1].$$

Since $g \overset{a.e}{=} 0$ almost everywhere on $\left[0, \frac{1}{2}\right)$, continuity of g implies that

$$g(x) = 0 \qquad \text{for every } x \in \left[0, \frac{1}{2}\right).$$

Indeed, if $g(x_0) \neq 0$ for some $x_0 < \frac{1}{2}$, then by continuity, there would be an open interval containing x_0 on which $g \neq 0$, contradicting the fact that $g = 0$ almost everywhere on $\left[0, \frac{1}{2}\right)$. Therefore,

$$g\left(\frac{1}{2}\right) = 0.$$

Similarly, since $g \overset{a.e}{=} 1$ almost everywhere on $\left(\frac{1}{2}, 1\right]$, continuity implies that

$$g(x) = 1 \qquad \text{for every } x \in \left(\frac{1}{2}, 1\right].$$

Consequently, by taking the right hand side limit at $x = \dfrac{1}{2}$. we get

$$g\left(\frac{1}{2}\right) = 1.$$

This is a contradiction. Therefore, a function may be continuous almost everywhere without being equal almost everywhere to any continuous function.

(b) No. Consider the function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} 0, & x \in [0, 1] \cap \mathbb{Q} \\ 1, & x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$

The density of the set of rationals and irrationals in $[0, 1]$ implies that f is not continuous anywhere on $[0, 1]$. Let $g : [0, 1] \to [0, 1]$ be defined by $g \equiv 1$. Then, g is continuous. Because the set of rational numbers is of measure zero, we have $f \overset{a.e}{=} g$ on $[0, 1]$, even though f is not continuous at all (let alone almost everywhere).

3. Let $f(x)$ be a function of bounded variation on $[a, b]$ and

$$v(x) = TV\left(f|_{[a,x]}\right)$$

be its total variation function ($TV(f) = v(b)$ is the total variation of f on $[a,b]$). Show that

(a)
$$|f'(x)| \leq v'(x)$$
for almost every $x \in [a,b]$ and that
$$\int_{[a,b]} |f'| \leq TV(f);$$

(b)
$$\int_{[a,b]} |f'| = TV(f)$$
if f in addition is absolutely continuous.

(a) *Proof.* Recall that
$$v(x) = TV\left(f|_{[a,x]}\right), \qquad x \in [a,b].$$
In particular, v is non-decreasing and $v(a) = 0$. For any $a \leq x < y \leq b$, we have
$$|f(y) - f(x)| \leq TV\left(f|_{[x,y]}\right) = v(y) - v(x).$$
Therefore,
$$-(v(y) - v(x)) \leq f(y) - f(x) \leq v(y) - v(x).$$
From the left inequality, we get
$$(v+f)(y) - (v+f)(x) = v(y) - v(x) + f(y) - f(x) \geq 0$$
and from the right inequality, we get
$$(v-f)(y) - (v-f)(x) = v(y) - v(x) - f(y) + f(x) \geq 0.$$
Thus, both $v+f$ and $v-f$ are non-decreasing functions on $[a,b]$.
Since every monotone function is differentiable almost everywhere, $v+f$ and $v-f$ are differentiable almost everywhere. Consequently, v and f are differentiable almost everywhere. The reason is:
v is differentiable almost everywhere because $v = \frac{1}{2}[(v+f) + (v-f)]$.
f is differentiable almost everywhere because $f = \frac{1}{2}[(v+f) - (v-f)]$.
At every point where these derivatives exist, we get
$$(v+f)'(x) = v'(x) + f'(x) \geq 0$$

and
$$(v-f)'(x) = v'(x) - f'(x) \geq 0.$$
Thus,
$$-v'(x) \leq f'(x) \leq v'(x),$$
and therefore
$$|f'(x)| \leq v'(x)$$
for almost every $x \in [a,b]$.

Because v is non-decreasing, its derivative is nonnegative almost everywhere, and
$$\int_a^b v'(x)\,dx \leq v(b) - v(a).$$
It follows that
$$\int_a^b |f'(x)|\,dx \leq \int_a^b v'(x)\,dx \leq v(b) - v(a) = TV(f).$$
Hence,
$$\int_{[a,b]} |f'| \leq TV(f).$$
This completes the proof of part (a). □

(b) *Proof.* Suppose in addition that f is absolutely continuous on $[a,b]$. By part (a),
$$\int_a^b |f'(x)|\,dx \leq TV(f).$$
To prove the reverse inequality, let
$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$
be an arbitrary partition of $[a,b]$. Since f is absolutely continuous, the Fundamental Theorem of Calculus gives
$$f(x_i) - f(x_{i-1}) = \int_{x_{i-1}}^{x_i} f'(x)\,dx \qquad \text{for each} \qquad i = 1, \cdots, n.$$
Therefore,
$$\begin{aligned}\sum_{i=1}^n |f(x_i) - f(x_{i-1})| &= \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} f'(x)\,dx \right| \\ &\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f'(x)|\,dx \\ &= \int_a^b |f'(x)|\,dx.\end{aligned}$$

Taking the supremum over all partitions P, we get

$$TV(f) \leq \int_a^b |f'(x)|\, dx.$$

Thus, by combining this with the inequality from part (a), we get

$$\int_{[a,b]} |f'| = TV(f).$$

This completes the proof of part (b),

$\square$

4. Let E be a measurable set with $m(E) < \infty$. Show that

$$\lim_{\delta \to 0} \frac{m(E \cap [x-\delta, x+\delta])}{2\delta} = 1$$

for almost every $x \in E$.

(In fact, the limit is equal to χ_E, the characteristic function of E, almost everywhere on $\mathbb{R}$.)

Proof. Let χ_E be the characteristic function of E. Since E is measurable and $m(E) < \infty$, we have

$$\chi_E \in L^1(\mathbb{R}).$$

Let

$$F(x) := \int_0^x \chi_E(t)\ dt = m(E \cap [0, x]).$$

Then, by the Lebesgue Differentiation Theorem,

$$F'(x) = \chi_E(x) \qquad \text{for almost every } x \in E.$$

We also have

$$F'(x) = \lim_{\delta \to 0} \frac{F(x+\delta) - F(x-\delta)}{2\delta}.$$

That is

$$F'(x) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} \chi_E(t)\ dt = \lim_{\delta \to 0} \frac{1}{2\delta} m(E \cap [x-\delta, x+\delta]).$$

Hence,

$$\lim_{\delta \to 0} \frac{1}{2\delta} m(E \cap [x-\delta, x+\delta]) = \chi_E(x) \qquad \text{for almost every } x \in E.$$

But $\chi_E(x) \equiv 1$ on E. Thus,

$$\lim_{\delta\to 0}\frac{1}{2\delta}m(E\cap[x-\delta,x+\delta])=1 \qquad \text{for almost every } x\in E.$$

□

5. Let f be an integrable function on $[0,1]$ such that $\log|f|$ is also integrable on $[0,1]$, where, as usual, log is the logarithm base e. Show that

$$\lim_{p\to 0^+}\int_{[0,1]}\frac{|f|^p-1}{p}=\int_{[0,1]}\log|f|.$$

Proof. For $p>0$, define $\phi:[0,\infty)\to\mathbb{R}$ by

$$\phi_p(x):=\frac{x^p-1}{p}.$$

Let $p\in(0,1)$.

If $x\in[0,1]$, then,

$$|\phi_p(x)|=\left|\frac{x^p-1}{p}\right|=\frac{1-x^p}{p}\le\frac{1-x}{p}<\frac{1+x}{p}.$$

If $x>1$, then

$$|\phi_p(x)|=\left|\frac{x^p-1}{p}\right|=\frac{x^p-1}{p}\le\frac{x-1}{p}<\frac{x+1}{p}.$$

Thus, for any $0<p<1$, we have

$$|\phi_p(x)|\le\frac{x+1}{p} \qquad \text{for every } x\in[0,\infty) \tag{4}$$

Because $f\in L^1([0,1])$, f is finite for almost every point on $[0,1]$. And because $\log|f|\in L^1([0,1])$, $f(x)\neq 0$ for almost every point on $[0,1]$. Thus, $0<|f(x)|<\infty$ for almost every $x\in[0,1]$.

By combining with Equation (4), for any $0<p<1$, we then get

$$|\phi_p(|f(x)|)|\le\frac{|f(x)|+1}{p} \qquad \text{for almost every } x\in(0,\infty). \tag{5}$$

Because $f\in L^1(\mathbb{R})$, this implies

$$\phi_p\circ|f|\in L^1(\mathbb{R}) \qquad \text{for any } 0<p<1.$$

notice that, for $x > 0$,

$$\lim_{p\to 0^+}\left(\frac{x^p-1}{p}\right) \overset{L'Hopital}{=} \lim_{p\to 0^+}\frac{d}{dp}\left(\frac{x^p-1}{p}\right)$$
$$= \lim_{p\to 0^+}\frac{x^p\ln x - 0}{1}$$
$$= \ln x.$$

Hence, for almost every $x > 0$, we have

$$\lim_{p\to 0^+}\left(\frac{|f|^p-1}{p}\right) = \ln|f|. \tag{6}$$

Thus, by Equations (5), (6), and the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{p\to 0^+}\int_0^1 \frac{|f|^p-1}{p} = \int_0^1 \ln|f|.$$

□

6. Let f be as in the previous problem. Show that

   (a)
   $$\exp\left\{\int_{[0,1]}\log|f|\right\} \le \|f\|_p = \left(\int_{[0,1]}|f|^p\right)^{1/p}$$
   for any $p \in (0,1]$;

   (b)
   $$\lim_{p\to 0^+}\|f\|_p \le \exp\left\{\int_{[0,1]}\log|f|\right\}$$
   (use Problem 5).

   (This problem shows that
   $$\lim_{p\to 0^+}\|f\|_p = \exp\left\{\int_{[0,1]}\log|f|\right\}.)$$

   (a) *Proof.* From the proof of the previous problem we have $\phi_p \circ |f| \in L^1([0,1])$ for $p \in (0,1)$.

   That is
   $$\int_0^1\left|\frac{|f|^p-1}{p}\right| < \infty \qquad \text{for } 0 < p < 1.$$

This implies $\int_0^1 |f|^p < \infty$, and therefore $f \in L^p([0,1])$ for $0 < p \leq 1$ (the case $p = 1$ comes from the hypothesis $f \in L^1([0,1])$).
(Note: $f \in L^p([0,1]) \iff |f|^p \in L^1([0,1])$.)
Let $p \in (0,1]$. Because $\exp(x)$ is convex, and

$$\exp\left(\ln\left(|f|^p\right)\right) = |f|^p \in L^1([0,1]),$$

by Jensen's inequality we get

$$\exp\left(\int_0^1 \ln\left(|f|^p\right)\right) \leq \int_0^1 \exp\left(\ln\left(|f|^p\right)\right).$$

That is

$$\exp p\left(\int_0^1 \ln|f|\right) \leq \int_0^1 |f|^p.$$

This implies

$$\exp\left(\int_0^1 \ln|f|\right) \leq ||f||_p.$$

□

(b) From the previous problem, we have

$$\int_{[0,1]} \phi_p(|f|) = \int_{[0,1]} \frac{|f|^p - 1}{p} = \frac{1}{p}\left(\int_{[0,1]} |f|^p - 1\right).$$

Thus,

$$\int_{[0,1]} |f|^p = p\int_{[0,1]} \phi_p(|f|) + 1,$$

and therefore

$$||f||_p = \left(p\int_{[0,1]} \phi_p(|f|) + 1\right)^{\frac{1}{p}}.$$

By taking ln from both sides we get,

$$\ln||f||_p = \frac{1}{p}\ln\left(p\int_{[0,1]} \phi_p(|f|) + 1\right).$$

We have $\ln(1+x) \leq x$ for $x > 0$. Hence,

$$\begin{aligned}
\ln||f||_p &= \frac{1}{p}\ln\left(p\int_{[0,1]} \phi_p(|f|) + 1\right) \\
&\leq \frac{1}{p}\cdot p\int_{[0,1]} \phi_p(|f|) \\
&= \int_{[0,1]} \phi_p(|f|)
\end{aligned}$$

From the previous problem, we have

$$\lim_{p\to 0^+}\int_{[0,1]}\phi_p(|f|)=\int_{[0,1]}\ln|f|.$$

Thus,

$$\lim_{p\to 0^+}\ln||f||_p\leq\int_{[0,1]}\ln|f|.$$

From the continuity of ln we have

$$\ln\left(\lim_{p\to 0^+}||f||_p\right)\leq\int_{[0,1]}\ln|f|.$$

Because $\exp x$ is an increasing function, we then get

$$\lim_{p\to 0^+}||f||_p\leq\exp\left(\int_{[0,1]}\ln|f|\right).$$

This completes the proof of part (b).

JANUARY 2024

Below $m^*()$ and $m(\cdot)$ are the outer Lebesgue measure and the Lebesgue measure on the real line.

Problem 1. Let $\{E_n\}_n$ be pairwise disjoint measurable sets, $E = \bigcup E_n$, and A be any set. Show that

$$m^*(E \cap A) = \sum_n m^*(E_n \cap A).$$

Proof. By the countable sub-additivity,

$$m^*(E \cap A) \leq \sum_n m^*(E_n \cap A).$$

Thus, if we can show the opposite inequality, then we are done. Because E_1 is measurable, by the definition of measurability and the pairwise disjointness of $\{E_n\}_{n=1}$ we get

$$m^*(E \cap A) = m^*(E_1 \cap A) + m^*\Big(\bigsqcup_{n=2}(E_n \cap A)\Big).$$

This implies for any $N \in \mathbb{N}$,

$$m^*(E \cap A) = \sum_{n=1}^{N} m^*(E_n \cap A) + m^*\Big(\bigsqcup_{n=N+1}(E_n \cap A)\Big).$$

Therefore, for any $N \in \mathbb{N}$,

$$m^*(E \cap A) \geq \sum_{n=1}^{N} m^*(E_n \cap A).$$

Then, by taking the limit as $N \to \infty$, we get

$$m^*(E \cap A) \geq \sum_{n=1} m^*(E_n \cap A).$$

This completes the proof. $\square$

Problem 2. Let f be a continuous function on $[0,1]$. Show that there exists a measurable subset $E \subset [0,1]$ such that $f(E)$ is not measurable if and only if there exists $A \subset [0,1]$ such that $m(A) = 0$ and $m^*(f(A)) > 0$.

Proof. $(\Rightarrow)$. Suppose there is a measurable set $E \subset [0,1]$ such that $f(E)$ is not measurable. Because E is measurable, there is a G_δ set $G \supset E$ such that $m(G \setminus E) = 0$. Let $A := G \setminus E$. So, $m(A) = 0$. Because the continuous image of a G_δ set is measurable, and sets with zero outer-measure are also measurable, we must then have $f(A)$ non-measurable. Because

$$f(G) = f(E) \cup f(A),$$

if $f(E)$ is non-measurable, then $f(A)$ cannot be measurable. Hence, $m^*(f(A)) > 0$.

$(\Leftarrow)$ Suppose there exists $A \subset [0,1]$ such that $m(A) = 0$ and $m^*(f(A)) > 0$. Then by Vitali's theorem, there is $F \subset f(A)$ such that F is non-measurable. Let $E = A \cap f^{-1}(F)$. Then E is measurable as $m(E) = 0$ and $f(E) = F$ is non-measurable. $\square$

Problem 3. Let $\{f_n\}_n$ be a sequence of measurable functions on E, $m(E) < \infty$. Assume that for every $x \in E$ there exists a constant M_x such that $|f_n(x)| \leq M_x$ for all n. Show that for every $\epsilon > 0$ there exists a closed set $F_\epsilon \subseteq E$ and a constant M_ϵ such that $|f_n(x)| \leq M_\epsilon$, for all n and $x \in F_\epsilon$, and $m(E \backslash F_\epsilon) < \epsilon$.

Proof. For each $M \in \mathbb{N}$, define

$$E_M = \{x \in E : |f_n(x)| \leq M \text{ for every } n \in \mathbb{N}\}.$$

Equivalently,

$$E_M = \bigcap_{n=1}^{\infty} \{x \in E : |f_n(x)| \leq M\}.$$

Since each f_n is measurable, E_M is measurable. Moreover, we have

$$E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq E_{n+1} \subseteq \cdots.$$

By hypothesis, for every $x \in E$ there exists a finite constant M_x such that

$$|f_n(x)| \leq M_x \qquad \text{for every } n \in \mathbb{N}.$$

By choosing an integer $M \geq M_x$, we obtain $x \in E_M$. Hence,

$$E = \bigcup_{M=1}^{\infty} E_M.$$

Because $m(E) < \infty$, continuity from below gives

$$m(E_M) \nearrow m(E) \qquad \text{as } M \to \infty.$$

Therefore, there exists $M_\varepsilon \in \mathbb{N}$ such that

$$m(E \setminus E_{M_\varepsilon}) < \frac{\varepsilon}{2}.$$

By the inner regularity of Lebesgue measure, there exists a compact set $F_\varepsilon \subseteq E_{M_\varepsilon}$ such that

$$m(E_{M_\varepsilon} \setminus F_\varepsilon) < \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned} m(E \setminus F_\varepsilon) &\leq m(E \setminus E_{M_\varepsilon}) + m(E_{M_\varepsilon} \setminus F_\varepsilon) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Finally, since $F_\varepsilon \subseteq E_{M_\varepsilon}$, we have

$$|f_n(x)| \leq M_\varepsilon$$

for every $n \in \mathbb{N}$ and every $x \in F_\varepsilon$. Since F_ε is compact, it is closed. This completes the proof. $\square$

Proof 2.

Proof. Let $\epsilon > 0$ be given. Because $m(E) < \infty$, we can find $N_\epsilon \in \mathbb{N}$ such that $m(E \backslash [-N_\epsilon, N_\epsilon]) < \epsilon/2$. Let $G_\epsilon := E \cap [-N_\epsilon, N_\epsilon]$. Then, by Lusin's theorem, we can find a decreasing sequence $(F_n)_{n=1}$ of compact subsets of G_ϵ such that for each $n \in \mathbb{N}$, f_n is continuous on F_n and $m(F_\epsilon \backslash F_n) \leq \epsilon \sum_{k=1}^{n} 1/2^{k+1}$. Let $F_\epsilon = \bigcap_{n=1} F_n$. Then $m(E \backslash F_\epsilon) \leq \epsilon/2 + \epsilon \sum_{k=1} 1/2^{k+1} = \epsilon/2 + \epsilon/2 = \epsilon$.

It remains to show that there is a uniform constant M_ϵ such that $|f_n(x)| \leq M_\epsilon$, for all n and $x \in F_\epsilon$. For a contradiction let's assume that it is not the case. Then the decreasing sequence $(A_j)_{j=1}$ of compact sets defined by

$$A_j := \overline{\{x \in F_\epsilon : |f_{n_j}(x)| \geq j \text{ for some } n_j \in \mathbb{N}\}}$$

has non-empty intersection by the Cantor's intersection theorem. Then for any $x_0 \in \bigcap_{j=1} A_j$, we have $M_{x_0} \geq |f_{n_j}(x_0)| \geq j$ for all $j \in \mathbb{N}$. This implies $M_{x_0} = \infty$. Contradiction. $\square$

Problem 4. Let $\{f_n\}_n$ be sequence of functions on $[0, 1]$ whose total variations are uniformly bounded. If $f_n \to f$ pointwise on $[0,1]$ as $n \to \infty$, show that f is a function of bounded variation. Will the conclusion remain true if we simply assume that each f_n is a function of bounded variation? (justify your answer)

Proof. Let $0 < \epsilon < 1$ be given. Let $\mathcal{P} := \{0 = x_0, x_1 \cdots x_{m-1} = 1\}$ be a partition of $[0,1]$. Then for each $j \in \{0, \cdots m-1\}$, there is f_{n_j} such that $|f_{n_j}(x_j) - f(x_j)| < \frac{\epsilon}{2m}$. Let $M \geq \max\{n_j : j = 0, \cdots, m-1\}$. Then $|f_M(x_j) - f(x_j)| < \frac{\epsilon}{2m}$ for all $j = 0, \cdots, m-1$. Let $K > 0$ be a uniform bound such that $TV(f_n) < K$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} V(f,\mathcal{P}) &= \sum_{j=1}^{m-1} |f(x_j) - f(x_{j-1})| \\ &\leq \sum_{j=1}^{m-1} |f(x_j) - f_M(x_j)| + \sum_{j=1}^{m-1} |f_M(x_j) - f_M(x_{j-1})| + \sum_{j=1}^{m-1} |f_M(x_{j-1}) - f(x_{j-1})| \\ &\leq \epsilon/2 + K + \epsilon/2 = \epsilon + K < 1 + K. \end{aligned}$$

Hence $f \in \mathcal{F}_{BV}([0,1])$. □

For the second part: No. Let f be any (almost everywhere) continuous function on $[0,1]$ which is not of bounded variation. By Weierstrass polynomial approximation theorem, there is a sequence $(P_n)_{n=1}$ of polynomial functions on $[0,1]$ that converges uniformly to f. But each P_n is of bounded variation as any polynomial on a compact interval is Lipschitz.

Problem 5. Let $\{f_n\}_n$ be sequence of measurable functions on a measurable set E that converges almost everywhere on E to an integrable function f. Show that

$$\int_E |f_n - f| \to 0 \text{ if and only if } \int_E |f_n| \to \int_E |f|$$

(both limits taking place as $n \to \infty$).

Proof. ($\Rightarrow$). Because $\big||f_n| - |f|\big| \leq |f_n - f|$, we have

$$\int_E |f| - \int_E |f_n - f| \leq \int_E f_n \leq \int_E |f| + \int_E |f_n - f|.$$

. Then, by taking the limit, we get

$$\lim_{n\to\infty} \int_E |f_n| = \int_E |f|.$$

Note: By the second inequality and the hypothesis, we have that for all but finitely many $n \in \mathbb{N}$, $f_n \in L^1([0,1])$.

($\Leftarrow$). Notice that $|f_n - f| \to 0$, and $\int_E |f_n - f| \leq \int_E |f_n| + \int_E |f| < \infty$ for all but finitely many $n \in \mathbb{N}$ ($\because$ $\int_E |f_n| \to \int_E |f|$). Then, by the General Lebesgue Dominated Convergence Theorem, we get the claim $\int_E |f_n - f| \to 0$. $\square$

Problem 6. Let $f \in L^p[0,1]$ for some $p > 1$. Show that $f \in L^1[0,1]$ and that for any $c \in (0, ||f||_1)$ it holds that

$$m(\{x \in [0,1] : |f(x)| > c\}) \geq \Big(\frac{||f||_1 - c}{||f||_p}\Big)^q,$$

where q is the conjugate exponent of p.

Proof. Let $E_1 := \{x \in [0,1] : |f(x)| > 1\}$. Then

$$||f||_1 \leq \int_{E_1} |f| + \int_{[0,1]\setminus E_1} |f| \leq ||f||_p + 1 < \infty.$$

Thus $f \in L^1[0,1]$.

If $||f||_p = 0$, then $f \overset{a.e}{=} 0$. Thus, we have the claim vacuously, as such a $0 < c < 0$ does not exist. So, suppose $||f||_p \neq 0$. Because $c \in (0, ||f||_1)$, $||f||_1 - c > 0$. Let $E_c := \{x \in [0,1] : |f(x)| > c\}$. Then

$$||f||_1 \leq c + \int_{[0,1]} |f| \cdot \chi_{E_c} \leq c + ||f||_p \cdot m(E_c)^{1/q}.$$

Hence

$$m(E_c) \geq \Big(\frac{||f||_1 - c}{||f||_p}\Big)^q.$$

$\square$

AUGUST 2023

In what follows, $m(\cdot)$ is the Lebesgue measure on the real line and E is measurable.

Problem 1. Let E be a measurable set for which there exists $\delta \in (0,1)$ such that $m(E \cap I) > \delta m(I)$ for any interval $I \subset (-\infty, \infty)$. Show that $m((-\infty,\infty)\backslash E) = 0$.

Proof. Let $F \subset (-\infty,\infty)\backslash E$ be such that $m(F) < \infty$. Then for any $\epsilon > 0$, there is a countable collection $\{I_n\}_{n=1}$ of disjoint open intervals such that $\bigsqcup_{n=1} I_n \supset F$ and $\sum_{n=1} m(I_n) < m(F) + \delta\epsilon$. Hence

$$\delta \sum_{n=1} m(I_n) < m(E \cap \bigsqcup_{n=1} I_n) < m(\Big(\bigsqcup_{n=1} I_n\Big)\backslash F) = \sum_{n=1} m(I_n) - m(F) = \delta\epsilon.$$

Thus, $\sum\limits_{n=1} m(I_n) < \epsilon$. Since $\epsilon > 0$ is arbitrary, this implies $m(F) = 0$ as $F \subset \bigsqcup_{n=1} I_n$. Because F is arbitrary, and any measurable set with infinite measure always have a measurable subset with finite measure, we have the claim. $\square$

Problem 2. A sequence of measurable functions f_n on E converges to a measurable function f *in measure* on E if $\lim_{n\to\infty} m\{x : |(f_n - f)(x)| > \delta\} = 0$ for any $\delta > 0$. Show that

(a) almost everywhere convergence implies convergence in measure if $m(E) < \infty$;

(b) almost everywhere convergence does not imply in general convergence in measure if $m(E) = \infty$;

(c) converge in measure does not imply in general almost everywhere convergence even if $m(E) < \infty$.

Proof. (a) Suppose $f_n \overset{a.e}{\to} f$ on E. Let $\epsilon > 0$. Then by Egoroff's theorem, there is closed

$F \subset E$ such that $f_n \rightrightarrows f$ on F and $m(E\backslash F) < \epsilon$. Therefore for any $\delta > 0$, we have

$$\begin{aligned} m\{x \in E : |(f_n - f)(x)| > \delta\} &= m\{x \in F : |(f_n - f)(x)| > \delta\} + \\ &\quad m\{x \in E\backslash F : |(f_n - f)(x)| > \delta\} \\ &\leq m\{x \in F : |(f_n - f)(x)| > \delta\} + \epsilon. \end{aligned}$$

Because $\epsilon > 0$ is arbitrary and $\lim_{n\to\infty} m\{x \in F : |(f_n - f)(x)| > \delta\} = 0$, we have the claim.

(b) Let $E = [0, \infty)$ and define the sequence of measurable functions $(f_n)_{n=1}$ on E by $f_n = \chi_{[n,\infty)}$. Thus for any $x \in E$, $f_n(x) \to 0$. But, for example,

$$\lim_{n\to\infty} m\{x : |(f_n - 0)(x)| > 1/2\} = \lim_{n\to\infty} m([n, \infty)) = \infty.$$

(c) Define a collection $\{E_{k,m}\}_{0\leq k\leq m-1,\ m=1}$ of intervals of $[0, 1]$ by $E_{k,m} := [k/m, (k+1)/m]$. Define the sequence $(f_n)_{n=0}$ of measurable functions on $[0, 1]$ by $f_n = \chi_{E_{k,m}}$ where $n = m(m-1)/2 + k$, $n \in \mathbb{N}$, $0 \leq k \leq m-1$. In other words, $(f_n)_{n=1}$ are the indicator functions of the sequence of sets:

$$\Big([0, 1], [0, \frac{1}{2}], [\frac{1}{2}, 1], [0, \frac{1}{3}], [\frac{1}{3}, \frac{2}{3}], [\frac{2}{3}, 1], \cdots\Big).$$

Then for any $x \in [0, 1]$, we can find sub sequences $(f_{n_i})_{i=1}$ and $(f_{n_j})_{j=1}$, say, such that $f_{n_i}(x) = 0$ and $f_{n_j}(x) = 1$ (simply by choosing k, m so that $x \notin E_{k,m}$ and $x \in E_{k,m}$ respectively). Thus $(f_n)_{n=1}$ does not converge point-wise anywhere in $[0, 1]$. But for any $0 < \delta < 1$ we have

$$m\big(\{x \in [0, 1] : f_n(x) > \delta\}\big) = m(E_{k,m}) = 1/m < 1/\sqrt{n},$$

and for any $\delta \geq 1$ we have $m\big(\{x \in [0, 1] : f_n(x) > \delta\}\big) = 0$. Thus $(f_n)_{n=1}$ converges to $f \equiv 0$ in measure on $[0, 1]$.

□

See the following *Problem* 3 for a slightly different example.

Problem 3. Show that if functions f_n are non-negative and measurable on E and $\int_E f_n \leq 1/n^2$, then $f_n \to 0$ holds almost everywhere on E. Will the claim remain true if $1/n^2$ is replaced by $1/n$?

Proof. (See Problem 5 of August 2021 for a different proof)

Let $F := \{x \in E : f_n(x) \nrightarrow 0\}$. Then

$$F = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x \in E : f_n(x) > 1/k\}.$$

With Chebyshev's inequality and the continuity of the Lebesgue measure, we have

$$\begin{aligned}
m(F) &= m\big(\bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \{x \in E : f_n(x) > 1/k\}\big) \\
&\leq \sum_{k=1}^{\infty} \lim_{N\to\infty} m\big(\bigcup_{n=N}^{\infty} \{x \in E : f_n(x) > 1/k\}\big) \\
&\leq \sum_{k=1}^{\infty} \lim_{N\to\infty} \sum_{n=N}^{\infty} m\big(\{x \in E : f_n(x) > 1/k\}\big) \\
&\leq \sum_{k=1}^{\infty} \lim_{N\to\infty} \sum_{n=N}^{\infty} k \int_E f_n \leq \sum_{k=1}^{\infty} \lim_{N\to\infty} \sum_{n=N}^{\infty} \frac{k}{n^2} = 0.
\end{aligned}$$

□

No. We can inductively construct a sequence $(a_n)_{n=0}$ of numbers in $[0,1]$ with the following properties:

i. $a_0 = 0$ and for any $n \in \mathbb{N}$, $a_n \neq a_{n+1}$.

ii. For $n \in \mathbb{N}$, if $a_n + 1/n \leq 1$, then let $a_{n+1} := a_n + 1/n$.

iii. For $n \in \mathbb{N}$, if $a_n < 1$ and $a_n + 1/n > 1$, then let $a_{n+1} = 1$ and $a_{n+2} = 0$.

So the sequence has an infinite number of finite strings of consecutive terms starting from 0 and ending at 1, again and again. This is possible as $\sum_{n=1} 1/n = \infty$ implies that for any $k \in \mathbb{N}$, there is $K \in \mathbb{N}$ such that $\sum_{n=k}^{K} 1/n \geq 1 = m([0,1])$.

Then we can define a sequence $I_n := [b_n, b_{n+1}]$ of closed sub-intervals of $[0,1]$ where $b_n = a_{n+K_n}$ for some $K_n \in \mathbb{N}$ such that $b_{n+1} - b_n \leq 1/n$, and for any $n \in \mathbb{N}$, there exist $M_n \leq n < N_n$ such that $[0,1] = \bigcup_{n=M_n}^{N_n} I_n$. Here are the first few intervals of such a sequence:

$$\begin{gathered}
[0,1], \\
\Big[0, \frac{1}{2}\Big], \Big[\frac{1}{2}, \frac{5}{6}\Big], \Big[\frac{5}{6}, 1\Big] \\
\Big[0, \frac{1}{5}\Big], \Big[\frac{1}{5}, \frac{11}{30}\Big], \Big[\frac{11}{30}, \frac{107}{210}\Big], \cdots
\end{gathered}$$

Therefore, for any $x \in [0,1]$, we can find two sub-sequences $(I_{n_i})_{i=1}$ and $(I_{n_j})_{j=1}$ such that $x \in I_{n_i}$ for all $i \in \mathbb{N}$ and $x \notin I_{n_j}$ for all $j \in \mathbb{N}$. Hence, the sequence $(f_n)_{n=1}$ of functions on

$[0,1]$ defined by $f_n := \chi_{I_n}$ does not converge pointwise anywhere in $[0,1]$. But

$$\int_{[0,1]} |f_n| = m(I_n) \leq \frac{1}{n}.$$

Problem 4. Let f be an integrable function on a bounded measurable set E. Find $\lim_{p\to 0^+} \int_E |f|^p$.

Answer. We prove

$$\lim_{p\to 0^+} \int_E |f|^p = m(E\backslash E^*).$$

Here $E^* := f^{-1}(\{0,\infty\})$.

Let $E_1 := \{x \in E : 0 < |f(x)| \leq 1\}$ and $E_2 := \{x \in E : \infty > |f| > 1\}$. Then for all $0 < p < 1$, $|f\chi_{E_1}|^p \leq 1$, and $\lim_{p\to 0^+} |f\chi_{E_2}(x)|^p = 1$. Because $f \in L^1(E)$, $m(f^{-1}(\{\infty\})) = 0$. Thus, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{p\to 0^+} \int_E |f|^p = \lim_{p\to 0^+} \int_{E_1} |f|^p + \lim_{p\to 0^+} \int_{E_2} |f|^p = m(E_1) + m(E_2) = m(E\backslash E^*).$$

Q: What if E is an unbounded measurable set with finite measure?

An idea: $E = \bigcup_{n=1}^{\infty} E_n$, where $E_n := E \cap [n, n+1]$.

Problem 5. Let $E \subset [0,1]$ be a Borel set such that $0 < m(E\cap I) < m(I)$ for any subinterval of $[0,1]$. Show that $f(x) = m([0,x] \cap E) - m([0,x]\backslash E)$ is absolutely continuous on $[0,1]$ but it is not monotone on any subinterval of $[0,1]$.

Proof. Because $f(x) = \int_0^x (\chi_E - \chi_{E^c})$, an indefinite integral, $f \in AC[0,1]$ by the fundamental theorem of Lebesgue integral calculus.

<u>For the second part:</u>

• **Proof 1:** The hypothesis "$0 < m(E \cap I) < m(I)$ *for any subinterval of* $[0,1]$" states that E is dense in $[0,1]$ and does not contain any intervals (hence it is totally disconnected). This, together with the fact "$f'(x) = \chi_E - \chi_{E^c}$ for almost every $x \in [0,1]$", imply that for any subinterval $J \subset [0,1]$, there exist points $a, b \in J$ such that $f'(a) = 1 > 0$ and $f'(b) = -1 < 0$. Thus f cannot be monotone on J.

• **Proof 2.** Notice that, because any Borel set is Lebesgue measurable,

$$f(x) = 2m([0,x] \cap E) - x.$$

<u>Case 1:</u> Suppose there is an interval J on which f is monotonically decreasing. Then for any $x_1, x_2 \in J$, $x_1 < x_2$ we have $f(x_1) - f(x_2) \geq 0$. Which implies $(x_2 - x_1)/m([x_1, x_2] \cap E) \geq 2$. For any given $\epsilon > 0$, choose $\{I_k\}_{k=1}$, $I_k \subset J$ so that $\bigsqcup_{k=1} I_k \supset E \cap J$ and $\sum_{k=1} m(I_k) < m(E \cap J) + \epsilon$. Then

$$m(E\cap J)+\epsilon > \sum_{k=1} m(I_k) = \sum_{k=1} \frac{m(I_k)}{m(I_k \cap E \cap J)} m(I_k \cap E \cap J) \geq 2 \sum_{k=1} m(I_k \cap E \cap J) = 2m(E \cap J).$$

Because $\epsilon > 0$ is arbitrary, this implies $m(E \cap J) = 0$. Contradiction.

<u>Case 2:</u> Suppose there is an interval J on which f is strictly increasing. Then for any $x_1, x_2 \in J$, $x_1 < x_2$ we have $f(x_2) - f(x_1) > 0$. Which implies $m([x_1, x_2] \cap E) > (x_2 - x_1)/2$. Then by *Problem* 1 (replacing $(-\infty, \infty)$ with J), $m(J \backslash E) = 0$. Which implies $m(J \cap E) = m(J)$. Contradiction.

Because J is an arbitrary interval, f is not monotone on any sub-interval of $[0, 1]$.

• **Proof 3.** If you are allowed, use the Lebesgue Density Theorem. $\square$

Q: Can E be a Lebesgue measurable but non-Borel set (with positive measure, of course)?

Q: Does there exist a Lebesgue measurable but non-Borel set $E \subset [0, 1]$ such that for any interval $I \subset [0, 1]$, $E \cap I$ is non-Borel?

An idea: Any Lebesgue measurable set is a Borel set minus a set of measure zero.

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Problem 1. Let A be a set such that $A \cap K \neq \emptyset$ for any compact subset K of real line of positive measure. Show that A has infinite outer Lebesgue measure.

Proof. If $m(A) < \infty$, then $m(A^c) > 0$. Hence by the inner regularity of the Lebesgue measure there is a compact set $K \subset A^c$, say, with $m(K) > 0$. This contradicts the hypothesis as $A \cap K = \emptyset$. □

Problem 2. Show that if f is continuous almost everywhere on $[0, 1]$, then it is measurable.

Proof. Let $E \subset [0, 1]$ be the set of all points at where f is continuous. Then $m(E^c) = 0$, and therefore E and fE ($\because$ the inverse image of any open set under fE is the intersection between an open set and E) are measurable. Because $f^{-1}((a, \infty)) \overset{a.e}{=} f|_E^{-1}((a, \infty))$ for any $a \in \mathbb{R}$, f is measurable. □

Problem 3. For a non-negative measurable function f on a measurable set E define

$$m_f(t) = m(\{f > t\}), t > 0,$$

where m is the Lebesgue measure. Show that $\int_E \varphi = \int_0^\infty m_\varphi$ for any non-negative simple function φ.

Proof. Let $\varphi = \sum_{j=1}^n c_k \chi_{E_j}$ with $c_1 < \cdots < c_n$. Then

$$m_\varphi(t) = \chi_{[0,c_1)}(t) \cdot \sum_{j=1}^n m(E \cap E_j) + \sum_{k=1}^{n-1} \Big[\chi_{[c_k,c_{k+1})}(t) \cdot \sum_{j=k+1}^n m(E \cap E_j)\Big].$$

Thus

$$\begin{aligned}\int_0^\infty m_\varphi &= c_1 \sum_{j=1}^n m(E \cap E_j) + \sum_{k=1}^{n-1} \Big[(c_{k+1} - c_k) \sum_{j=k+1}^n m(E \cap E_j)\Big] \\ &= \sum_{j=1}^n c_j \cdot m(E \cap E_j) = \int_E \varphi.\end{aligned}$$

□

Problem 4. In the setting of the previous problem and assuming its statement is true, show that $\int_E f = \int_0^\infty m_f$ for any non-negative measurable function f.

Proof. By the simple approximation theorem, we can find an increasing sequence $(\varphi_n)_{n=1}$ of positive simple functions such that $\varphi_n \nearrow f$ on E. By the monotone convergence theorem $\int_E \varphi_n \nearrow \int_E f$, and by the previous problem $\int_E \varphi_n = \int_0^\infty m_{\varphi_n}$ for all $n \in \mathbb{N}$. Thus if we can show $\lim_{n\to\infty} \int_0^\infty m_{\varphi_n} = \int_0^\infty m_f$, then we are done.

Notice that for any $t > 0$, $x \in \{x \in \mathbb{R} : f(x) > t\}$ if and only if $x \in \{x \in \mathbb{R} : \varphi_n(x) > t\}$ for all but finitely many $n \in \mathbb{N}$. Thus

$$\{x \in \mathbb{R} : f(x) > t\} = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{x \in \mathbb{R} : \varphi_n(x) > t\}.$$

Then

$$\begin{aligned}
m_f(t) &:= m(\{x \in \mathbb{R} : f(x) > t\}) \\
&= m\Big(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{x \in \mathbb{R} : \varphi_n(x) > t\} \Big) \\
&= \lim_{k\to\infty} m\Big(\bigcap_{n=k}^{\infty} \{x \in \mathbb{R} : \varphi_n(x) > t\} \Big), \because \text{continuity of Lebesgue measure} \\
&= \lim_{k\to\infty} m(\{x \in \mathbb{R} : \varphi_k(x) > t\}), \because \bigcap_{n=k}^{\infty} \{x \in \mathbb{R} : \varphi_n(x) > t\} = \{x \in \mathbb{R} : \varphi_k(x) > t\} \\
&= \lim_{n\to\infty} m(\{x \in \mathbb{R} : \varphi_n(x) > t\}) = \lim_{n\to\infty} m_{\varphi_n}(t).
\end{aligned}$$

Because $t > 0$ is arbitrary and $(m_{\varphi_n})_{n=1}$ is an increasing sequence of measurable functions, by the monotone convergence theorem we get

$$\lim_{n\to\infty} \int_0^\infty m_{\varphi_n} = \int_0^\infty m_f.$$

This completes the proof. □

Problem 5. Let f be integrable function of the real line. Show that $f = 0$ almost everywhere if $\int_I f = 0$ for any interval I of length 1.

Proof. For the sake of contradiction let's assume that there is $\epsilon > 0$ and a measurable set E_1^+ inside of an interval of length 1 such that $m(E_1^+) > 0$ and $f|E_1^+ > \epsilon$ (choose E_1^+ to be maximal, up to measure zero set, within a unit interval). Then the hypothesis implies that there is another set E_1^-, within an interval of length 1 to E_1^+, such that $m(E_1^+) = m(E_1^-)$ and $fE_1^- < -\epsilon$. By continuing in this way we can find a sequence $(E_n^+)_{n=1}$ of disjoint measurable sets with same (positive) measure such that $f|E_n^+ > \epsilon$ for all $n \in \mathbb{N}$. This violates the integrebility of f as

$$\int |f| > \int_{\bigsqcup_{n=1}^{\infty} E_n^+} f \geq \sum_{n=1}^{\infty} \epsilon \cdot m(E_n^+) = \epsilon \cdot \infty = \infty.$$

$\square$

Problem 6. Let $f \in L^p[0,1]$, $p \in (1, \infty)$, and F be an indefinite integral of f. Show that for each $x \in (0,1)$

$$\lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h^{1/q}} = 0,$$

where q is the conjugate exponent of p.

Proof. We have $F(x) = F(0) + \int_0^x f$ for all $x \in [0,1]$. Therefore, for any $x \in [0,1]$,

$$\begin{aligned} \lim_{h \to 0^+} \left| \frac{F(x+h) - F(x)}{h^{1/q}} \right| &\leq \lim_{h \to 0^+} \frac{\int_x^{x+h} |f|}{h^{1/q}} \\ &= \lim_{h \to 0^+} \frac{\int_{[0,1]} |f| \cdot \chi_{[x,x+h]}^2}{h^{1/q}} \\ &\overset{Holder}{\leq} \lim_{h \to 0^+} ||f \chi_{[x,x+h]}||_p. \ (*) \end{aligned}$$

Let $h = 1/n$, $n \in \mathbb{N}$. For fixed $x \in [0,1]$, define the sequence $(f_{x,n})_{n=1}$ of $L^1[0,1]$ functions by

$$f_{x,n}(t) := |f(t)|^p \cdot \chi_{[x,x+1/n]}(t).$$

Thus, $f_{x,n} \leq |f|^p$ for all $n \in \mathbb{N}$ and $f_{x,n}(t) \to 0$ for all $t \in [0,1] \backslash \{x\}$ as $n \to \infty$. Since the singletons have zero measure and $f^p \in L^1[0,1]$, the Lebesgue Dominated Convergence Theorem tells $\int_0^1 f_{x,n} \to 0$ for all $x \in [0,1]$. This, together with the squeeze theorem, gives us

$$\lim_{h \to 0^+} ||f \chi_{[x,x+h]}||_p = \lim_{n \to \infty} ||f \chi_{[x,x+\frac{1}{n}]}||_p = \lim_{n \to \infty} \left(\int_0^1 f_{x,n} \right)^{1/p} = 0. \qquad (**)$$

By $(*)$ and $(**)$ we have the claim. $\square$

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Problem 1. Let $f \to [0, \infty)$ be a measurable function and define its distribution function $d_f : [0.\infty) \to [0, \infty]$ defined by $d_f(t) = |\{x \in \mathbb{R} : f(x) > t\}|$.

i. Prove or disprove the statement that the function d_f is left continuous, that is to say

$$\lim_{t \nearrow t_0} d_f(t) = d_f(t_0), \text{ for each } t_0 > 0.$$

ii. Prove or disprove the statement that the function d_f is right continuous, that is to say

$$\lim_{t \searrow t_0} d_f(t) = d_f(t_0), \text{ for each } t_0 \leq 0.$$

Proof. i. Let $t_0 = 1$ and $f(x) = \chi_{[0,1)}(x)$. Then $d_f(t_0) = 0 \neq 1 = \lim_{t \nearrow t_0} d_f(t)$. So, $d_{\chi_{[0,1)}}$ is not left continuous.

ii. Let $t_0 \geq 0$. We prove $\lim_{t \searrow t_0} d_f(t) = d_f(t_0)$. For $t > t_0$, notice that

$$m(f^{-1}(t_0, \infty)) = m(f^{-1}(t_0, t]) + m(f^{-1}(t, \infty)) \quad (*)$$

Because $f^{-1}(t_0, t_1] \subset f^{-1}(t_0, t_2]$ for any $t_1 < t_2$, $\lim_{t \searrow t_0} m(f^{-1}(t_0, t])$ exists (continuity of Lebesgue measure). If it is 0, then we are done. If not, then there are $M > 0$ and $N > t_0$ such that $\lim_{t \searrow t_0} m(f^{-1}(t_0, t]) \geq M$ for all $t \in (t_0, N)$. Let $t \in (t_0, N)$. Because $t_0 \notin (t_0, t]$, there is a countable union of disjoint (can be de-generated) intervals $(I_k)_{k=1}^{\infty}$ in $(t_0, t]$ such that $m(f^{-1}(I_k)) \geq M$ for each k (let the enumeration begin from the right). This says $m(f^{-1}(t_0, t]) = \infty$. Because t is arbitrary, $\lim_{t \searrow t_0} m(f^{-1}(t, \infty)) \geq \lim_{n \to \infty} \sum_{k=1}^{n} m(I_k) \geq \lim_{n \to \infty} nM = \infty$. Here $\bigsqcup_{k=1}^{n} I_k \subset (t, N)$. So, $\lim_{t \searrow t_0} d_f(t) = \infty$. Then from $(*)$, $d_f(t_0) = \lim_{t \searrow t_0} d_f(t) = \infty$.

$\square$

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Show the following.

i. if $F \subset \mathbb{R}$ is an F_σ set then $f(F) \subset \mathbb{R}$ is an F_σ set.

i. if $F \subset \mathbb{R}$ is a measurable set then $f(F) \subset \mathbb{R}$ is a measurable set.

Proof. i . Because $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}[n, n+1]$, $F \subset \mathbb{R}$ is an F_σ set if and only if it is a countable union of compact sets. Since the continuous image of a compact set is compact and $f(A \cup B) = f(A) \cup f(B)$ for any sets (let's say measurable) $A, B \subset \mathbb{R}$, we have that $f(F)$ is an F_σ set if F is an F_σ set.

ii . Suppose $F \subset \mathbb{R}$ is a measurable set. Then $F = E \sqcup (F \backslash E)$ where E is an F_σ set and $m(F\backslash E) = 0$. Hence $f(F) = f(E) \cup f(F\backslash E)$. So, if we can show $m(f(F\backslash E)) = 0$, then by part (i) we have that F is measurable. In-particular, if we can show $m(f(A)) \leq m(A)$ for any measurable set A, then we are done. Because the semi-ring of bounded intervals generates the collection of all Lebesgue measurable sets (and with the fact that singletons have measure zero), it is enough to show $m(f([a, b])) \leq M(b - a)$ for any closed interval $[a, b]$. Here $M > 0$ is the Lipschitz constant of f. In-fact

$$m(f([a, b])) \leq \max_{x\in[a,b]} f(x) - \min_{x\in[a,b]} f(x) = f(\alpha) - f(\beta) \leq M(\alpha - \beta) \leq M(b - a).$$

Here $\alpha, \beta \in [a, b]$ are such that $f(\alpha) = \max_{x\in[a,b]} f(x)$ and $f(\beta) = \min_{x\in[a,b]} f(x)$.

□

Problem 3. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function. Show that the following are equivalent:

(*) for each measurable set $E \subset \mathbb{R}$, the set $\phi^{-1}(E)$ is measurable and $|\phi^{-1}(E)| = |E|$;

(**) for each measurable function $f : \mathbb{R} \to \mathbb{R}$, the function $f \circ \phi : \mathbb{R} \to \mathbb{R}$ is measurable and

$$\int_{\mathbb{R}} f \circ \phi = \int_{\mathbb{R}} f.$$

Proof. $(* \Rightarrow **)$ Since ϕ is measurable by the hypothesis, $f \circ \phi$ is measurable. Because $f : \mathbb{R} \to \mathbb{R}$ is measurable, there is a sequence $(\varphi_n)_{n=1}^\infty$ of simple functions that converges pointwise to f on $\mathbb{R}$ with $|\varphi_n| \leq |f|$ for all $n \in \mathbb{N}$ (if $f > 0$, choose $(\varphi_n)_{n=1}^\infty$ to be increasing). Therefore, if we can show the claim for any simple function, then we are done, as: If $f \in L^1(\mathbb{R})$, then use LDCT. If $f \notin L^1(\mathbb{R})$, then at-least one of $\int_{\mathbb{R}} f_-$ or $\int_{\mathbb{R}} f_+$ equals ∞. Then use the monotone convergence theorem to this f_- or f_+.

Let $f(x) = \sum_{k=1}^\infty a_k \chi_{E_k}(x)$ be a simple function. Then

$$\int_{\mathbb{R}} f \circ \phi = \sum_{k=1}^n a_k m(\phi^{-1}(E_k)) = \sum_{k=1}^n a_k m(E_k) = \int_{\mathbb{R}} f.$$

($* \Leftarrow **$) Letting f be the identity, we get $\phi : \mathbb{R} \to \mathbb{R}$ is measurable. And for any measurable set E, letting $f = \chi_E$, we get

$$m(\phi^{-1}(E)) = \int_{\mathbb{R}} f \circ \phi \overset{\text{Hypothesis}}{=} \int_{\mathbb{R}} f = m(E).$$

$\square$

Problem 4. Let $E \subset \mathbb{R}$ be a measurable set with $0 < |E| < \infty$. Let $f \in Ł^\infty(E)$.

i. Show that $f \in L^p(E)$ for each $p \in [1, \infty)$

ii. Show that $||f||_p \to ||f||_\infty$ as $p \to \infty$.

Proof. i. Let $p \in [1, \infty)$. Then $||f||_p \leq ||f||_\infty \cdot m(E)^{1/p} < \infty$. Thus $f \in L^p(E)$.

ii. Because $||f||_p \leq ||f||_q$ for any $p \leq q$, the limit $L := \lim_{p \longrightarrow \infty} ||f||_p$ exists in $[0, \infty]$. But since $f \in L^\infty(E)$, $L \in [0, \infty)$. Notice that $||f||_p = \left(\int_E |f|^p \right)^{1/p} \leq m(E)^{\frac{1}{p}} ||f||_\infty$. Then by taking the limit as $p \to \infty$ we get $L \leq ||f||_\infty$. To show the reverse inequality: for $n \in \mathbb{N}$, define

$$E_n := \{x \in E : ||f||_\infty \Big(1 - \frac{1}{n}\Big) \leq |f(x)| \leq ||f||_\infty\}.$$

Then $m(E_n) > 0$. (This is true as $m(E) > 0$ and $|| \cdot ||_\infty$ is the *essential sup norm*. If it is just the *sup norm*, then the claim is false. For a counter example consider $f : [0,1] \to [0,1]$ defined by $f(x) = 1$ if $x = 1/2$, and $f(x) = 0$ else). Then

$$||f||_p^p = \int_E |f|^p = \int_{E_n} |f|^p + \int_{E \setminus E_n} |f|^p \geq ||f||_\infty^p \cdot \Big(1 - \frac{1}{n}\Big)^p \cdot m(E_n)$$

Hence $L = \lim_{p \longrightarrow \infty} ||f||_p \geq \lim_{p \longrightarrow \infty} ||f||_\infty \cdot \left(1 - \frac{1}{n}\right) \cdot m(E)^{\frac{1}{p}} = ||f||_\infty \cdot \left(1 - \frac{1}{n}\right)$. Since this is true for any $n \in \mathbb{N}$, we get $L \geq ||f||_\infty$, and therefore the claim.

$\square$

Problem 5. Let $f \in L^1(\mathbb{R})$. For each $\lambda > 0$, put $f_\lambda(x) = \lambda f(\lambda x)$. Show that $f_\lambda \in L^1(\mathbb{R})$, and that $f_\lambda \to f$ in $L^1(\mathbb{R})$ as $\lambda \to 1$.

Proof. For $n \in \mathbb{N}$, we have $\int_{[-n,n]} |f_\lambda| = \int_{[-\lambda n, \lambda n]} |f| \leq ||f||_1 < \infty$. This is true for any $n \in \mathbb{N}$, $f_\lambda \in L^1(\mathbb{R})$. Define $g_\lambda := \frac{|f_\lambda| + |f| - |f_\lambda - f|}{2}$. Then $g_\lambda \geq 0$. By Lusin's theorem, for any $\epsilon > 0$, we can find a measurable set E_ϵ such that

(i). $||f \cdot \chi_{\mathbb{R} \setminus E_\epsilon}||_1 < \epsilon/2$,

(ii). $||f_\lambda \cdot \chi_{\mathbb{R} \setminus E_\epsilon}||_1 < \epsilon$ for all close enough λ to 1, and

(iii). $\lim_{\lambda \to 1} g_\lambda = |f|$ pointwise on E_ϵ.

Thus, by Fatou's lemma,

$$||f_{E_\epsilon}||_1 \leq ||f_{E_\epsilon}||_1 - \frac{1}{2} \limsup_{\lambda \to 1} ||(f_\lambda - f)_{E_\epsilon}||_1.$$

Hence $\lim_{\lambda \to 1} ||(f_\lambda - f)_{E_\epsilon}||_1 = 0$. This together with (ii) gives us

$$\lim_{\lambda \to 1} ||f_\lambda - f)||_1 < \epsilon.$$

Because this is true for any $\epsilon > 0$, we have the claim. $\square$

Problem 6. Assume $f_n \to f$ and $g_n \to g$ in $L^2(\mathbb{R})$. Show that $f_n g_n \to fg$ in $L^1(\mathbb{R})$.

Proof. We have that

$$\begin{aligned} ||f_n g_n - fg||_1 = ||(f_n - f)g_n + (g_n - g)f||_1 &\overset{\text{Minkowski}}{\leq} ||(f_n - f)g_n||_1 + ||(g_n - g)f||_1 \\ &\overset{\text{Holder}}{\leq} ||f_n - f||_2 \cdot \|g_n\|_2 + ||g_n - g||_2 \cdot \|f||_2. \end{aligned}$$

Because $g_n \to g$ in $L^2(\mathbb{R})$, $g_n^2 \to g^2$ in $L^1(\mathbb{R})$. Thus, by letting $n \to \infty$ in the above inequality, we get the claim as $f, g \in L^2(\mathbb{R})$. $\square$

AUGUST 2021

Problem 1. Let $n \in \mathbb{N}$ and $p_1, p_2, \cdots, p_n \in (1, \infty)$ be such that $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1$. If $f_1 \in L^{p_1}(\mathbb{R}), f_2 \in L^{p_2}(\mathbb{R}), \cdots, f_n \in L^{p_n}(\mathbb{R})$, then show that $f_1 f_2 \cdots f_n \in L^1(\mathbb{R})$.

Proof. We prove by induction on n. The base case (when $n = 2$) follows from Holder's inequality. Let $k \in \mathbb{N}_{\geq 2}$ and suppose that the claim is true for all $n = 2, \cdots, k$. That is; if $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} = 1$ and $f_1 \in L^{p_1}(\mathbb{R}), f_2 \in L^{p_2}(\mathbb{R}), \cdots, f_k \in L^{p_k}(\mathbb{R})$, then $f_1 f_2 \cdots f_k \in L^1(\mathbb{R})$.

Need to show if $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} + \frac{1}{p_{k+1}} = 1$ and $f_1 \in L^{p_1}(\mathbb{R}), f_2 \in L^{p_2}(\mathbb{R}), \cdots, f_k \in L^{p_k}(\mathbb{R}), f_{k+1} \in L^{p_{k+1}}(\mathbb{R})$, then $f_1 f_2 \cdots f_k f_{k+1} \in L^1(\mathbb{R})$. Let $q = \frac{p_k p_{k+1}}{p_k + p_{k+1}}$. Then $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{q} = 1$. This implies $q > 1$. Thus, if we can show $f_k f_{k+1} \in L^q(\mathbb{R})$, then we are done by the induction hypothesis as it implies $f_1 f_2 \cdots (f_k f_{k+1}) \in L^1(\mathbb{R})$. Notice that

$$f_k \in L^{p_k}(\mathbb{R}) \Rightarrow f_k^{p_k} \in L^1(\mathbb{R}) \Rightarrow f_k^q \in L^{\frac{p_k}{q}}(\mathbb{R}).$$

Similarly

$$f_{k+1} \in L^{p_{k+1}}(\mathbb{R}) \Rightarrow f_k^{p_{k+1}} \in L^1(\mathbb{R}) \Rightarrow f_{k+1}^q \in L^{\frac{p_{k+1}}{q}}(\mathbb{R}).$$

Since $\frac{q}{p_k} + \frac{q}{p_{k+1}} = 1$, by Holder's inequality, $||f_k^q f_{k+1}^q||_1 \leq ||f_k^q||_{\frac{p_k}{q}} + ||f_{k+1}^q||_{\frac{p_{k+1}}{q}} < \infty$. Hence $f_k f_{k+1} \in L^q(\mathbb{R})$. $\square$

Problem 2. Let A and B be Lebesgue measurable subsets of $\mathbb{R}$, with $A \subset B$. Show that $m(A) = m(B)$ if and only if every subset C satisfying $A \subset C \subset B$ is Lebesgue measurable.

Proof. ($\Rightarrow$) Suppose $m(A) = m(B)$. Then for any $C \subset \mathbb{R}$ with $A \subset C \subset B$, there is a set $E \subset \mathbb{R}$, say, of Lebesgue measure zero, such that $C = A \sqcup E$. Hence C is measurable.
($\Leftarrow$) For the contra-positive, assume $m(A) \lneq m(B)$. Then by Vitali's theorem, there is a non-measurable set $F \subset B \backslash A$. Then $C := A \sqcup F$ is non-measurable with $A \subset C \subset B$. $\square$

Problem 3.

(a). Prove that for every $\epsilon > 0$ there is an open subset of $[0, 1]$, whose closure is $[0, 1]$, and whose Lebesgue measure is less than ϵ.

(b). Prove that for every $a \in (0,1]$ there is an open subset of $[0,1]$, whose closure is $[0,1]$, and whose Lebesgue measure is equal to a.

Proof. (a). If $\epsilon > 1$, then take $(0,1)$ to be the open set. If $\epsilon \in (0,1]$, then take the set below in part (b) when, for example, $a = \frac{\epsilon}{2}$.

(b). Let $a \in (0,1]$. Consider the standard middle third Cantor set (let's denote it by $C_{\frac{1}{3}}$). Instead of removing middle thirds (the ratio of *"length of a removed interval in stage $n+1$"* to *"the length of the interval where it belonged as the middle third interval in stage n"* is $1:3$), let's remove the middle $(\frac{1+2a}{a})^{th}$. That is; the ratio of *"length of a removed interval in stage $n+1$"* to *"the length of the interval where it belonged as the middle $(\frac{1+2a}{a})^{th}$ interval in stage n"* is $1 : \frac{1+2a}{a}$. Let U_a be the disjoint union of those removed open intervals in this process. Hence U_a is open and clearly $\overline{U_a} = [0,1]$ as $[0,1]\backslash U_a$ does not contain any open interval. And

$$m(U_a) = \sum_{n=0}^{\infty} 2^n \Big(\frac{a}{1+2a}\Big)^{n+1} = a.$$

□

Problem 4. Let $f, g : [0,1] \longrightarrow \mathbb{R}$. Prove that

$$TV(fg) \le TV(f) \sup_{x\in[0,1]} |g(x)| + TV(g) \sup_{x\in[0,1]} |f(x)|,$$

where TV is the total variation. Give examples where the inequality is strict, and where the equality holds although none of the functions f, g is constant.

Proof. Let $\mathcal{P} = \{0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = 1\}$ be a partition of $[0,1]$. Then

$$\begin{aligned} V(fg, \mathcal{P}) &= \sum_{k=1}^{n} |f(a_k)g(a_k) - f(a_{k-1})g(a_{k-1})| \\ &\le \sum_{k=1}^{n} |f(a_k) - f(a_{k-1})||g(a_k)| + |g(a_k) - g(a_{k-1})||f(a_{k-1})| \\ &\le \sup_{x\in[0,1]} |g(x)| \sum_{k=1}^{n} |f(a_k) - f(a_{k-1})| + \sup_{x\in[0,1]} |f(x)| \sum_{k=1}^{n} |g(a_k) - g(a_{k-1})| \\ &= V(f,\mathcal{P}) \sup_{x\in[0,1]} |g(x)| + V(g,\mathcal{P}) \sup_{x\in[0,1]} |f(x)|. \end{aligned}$$

Then by taking the suprimum over all the partitions of $[0,1]$, we get the claim. □

For the strict inequality: Let $f(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}) \\ 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$ and $g(x) := \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ 0 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$. Then $fg = 0$.

Thus $TV(fg) = 0$ and $TV(f) \sup_{x\in[0,1]} |g(x)| + TV(g) \sup_{x\in[0,1]} |f(x)| = 1 \cdot 1 + 1 \cdot 1 = 2$.

For the equality: let $f(x)$ be as above and and let g be any function with $TV(g|_{[\frac{1}{2},1]}) = \infty$. Then the both sides of the inequality becomes ∞.

If need a finite equality: Let $f(x)$ be as above and $g(x) := \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$. Then $fg(x) := \begin{cases} 1 & \text{if } x = \frac{1}{2} \\ 0 & \text{else} \end{cases}$. Thus $TV(fg) = 2$ (for the partition, for example, $\{0 < \frac{1}{2} < 1\}$) and $TV(f) \sup_{x\in[0,1]} |g(x)| + TV(g) \sup_{x\in[0,1]} |f(x)| = 1 \cdot 1 + 1 \cdot 1 = 2$.

Problem 5. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $L^1([0,1])$, satisfying

$$\int_{[0,1]} |f_n| \leq \frac{1}{n^2}$$

for every $n \in \mathbb{N}$. Show that f_n converges to 0 pointwise a.e.

Proof. Let $\epsilon > 0$ be given. Then by Chebyshev's inequality,

$$m(\{x \in [0,1] : |f_n(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_{[0,1]} |f_n| \leq \frac{\epsilon}{n^2}.$$

Thus $\sum_{n=1}^{\infty} m(\{x \in [0,1] : |f_n(x)| \geq \epsilon\}) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{n^2} < \infty$. Therefore, by the Borel-Cantelli lemma, for almost all $x \in [0,1]$, there is $N \in \mathbb{N}$ such that for each $n > N$, $x \notin \{x \in [0,1] : |f_n(x)| \geq \epsilon\}$. In other words: for almost all $x \in [0,1]$ there is $N \in \mathbb{N}$ such that for each $n > N$, $|f_n(x)| < \epsilon$. Hence the claim. $\square$

Problem 6. Let $f : [0,1] \longrightarrow \mathbb{R}$ be a continuous function. Evaluate

$$\lim_{n\longrightarrow\infty} \int_0^1 nx^{n-1} f(x) dx.$$

Answer. For a fixed $a \in \mathbb{R}$ and $k \in \mathbb{N}$, we get $\lim_{n\longrightarrow\infty} \int_0^1 nx^{n-1} ax^k dx = a$. Thus, by the linearity of the Lebesgue integral, $\lim_{n\longrightarrow\infty} \int_0^1 nx^{n-1} P(x) dx = P(1)$ for any polynomial $P(x)$ on $[0,1]$. Let f be a continuous function on $[0,1]$. By the Weierstrass approximation theorem, there is a sequence $(P_m(x))_{m=1}^{\infty}$ of polynomials on $[0,1]$ such that $P_m \rightrightarrows f$. Because one of the limits is uniform, we can interchange the two limits. Thus,

$$\begin{aligned} f(1) = \lim_{m\longrightarrow\infty} P_m(1) &= \lim_{m\longrightarrow\infty}\left(\lim_{n\longrightarrow\infty}\int_0^1 nx^{n-1}P_m(x)dx\right) \\ &= \lim_{n\longrightarrow\infty}\left(\lim_{m\longrightarrow\infty}\int_0^1 nx^{n-1}P_m(x)dx\right) \\ &= \lim_{n\longrightarrow\infty}\int_0^1 \lim_{m\longrightarrow\infty} nx^{n-1}P_m(x)dx \\ &= \lim_{n\longrightarrow\infty}\int_0^1 nx^{n-1}f(x)dx. \end{aligned}$$

Note: We cannot start with $\lim_{n\longrightarrow\infty}\int_0^1 nx^{n-1}f(x)dx$ as we still don't know if the limit exists.

WINTER 2021

1. Give an example of a measurable set $E \subset \mathbb{R}$ of finite measure such that $m(E \cap (E+r)) > 0$ for every $r \in \mathbb{R}$ (where $E + r = \{x + r : x \in E\}$).

Example 1.

$$\bigsqcup_{n=1}^{\infty} [-n - \frac{1}{2^n}, -n] \sqcup [-1, 1] \sqcup \bigsqcup_{n=1}^{\infty} [n, n + \frac{1}{2^n}].$$

Example 2. Let $(q_n)_{n \in \mathbb{N}}$ be an enumeration of $\mathbb{Q}$. Then

$$\bigcup_{n=1}^{\infty} [q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n}].$$

2. Let $f_n : [0,1] \longrightarrow [0, \infty)$ $(n = 1, 2, 3, \cdots)$ be measurable functions. Prove that there exist numbers $a_n > 0$ $(n = 1, 2, 3, \cdots)$ such that the series $\sum_{n=1}^{\infty} a_n f_n(x)$ converges for almost every $x \in [0, 1]$.

Proof. Because f_n is finite, $\lim_{k \longrightarrow \infty} f_n^{-1}((k, \infty)) = 0$ for each $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, there is $\delta(n) > 1$, say, such that $m(E_n) \leq 1/n^2$. Here $E_n := f_n^{-1}((\delta(n), \infty))$. Let $a_n = \frac{1}{\delta(n) \cdot 2^n}$. Then the series $\sum_{n=1}^{\infty} a_n f_n(x)$ diverges if and only if x belongs to infinitely many $E_n's$. That is $x \in \bigcap_{k=1}^{\infty} E_{n_k}$. The Borel-Cantelli lemma says the set consisting of such points has measure zero as $\sum_{n=1}^{\infty} m(E_n) \leq \sum_{n=1}^{\infty} 1/n^2 < \infty$. □

3. Let $f \in L^1(\mathbb{R})$. Prove that for every measurable set $E \subset \mathbb{R}$ of finite measure we have

$$\lim_{t \longrightarrow \infty} \int_E f(x + t)\ dx = 0.$$

Proof. Let $\epsilon > 0$ be given. Because $f \in L^1(\mathbb{R})$ there is $\delta > 0$ such that for any $F \subset \mathbb{R}$ with $m(F) < \delta$ we get $\int_F |f| < \frac{\epsilon}{2}$. Because $m(E) < \infty$, we can find an interval $[-a, a]$, say, such that $\int_{\mathbb{R} \setminus [-a,a]} |f| < \frac{\epsilon}{2}$ and $m(E \setminus [-a, a]) < \min\{\delta, \frac{\epsilon}{2}\}$. Because Lebesgue measure is invariant under translation, for all $t > 2a$

$$|\int_E f(x+t)| \leq \int_E |f(x+t)| \leq \int_{E \cap [-a,a]} |f(x+t)| + \frac{\epsilon}{2} = \int_{(E \cap [-a,a])+t} |f(x)| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

4. Assume that for a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and a constant $C \in \mathbb{R}$, the total variation of f on every compact interval is smaller than or equal to C. Prove that

$$\int_{\mathbb{R}} |f(x+h) - f(x)| \; dx \leq C|h|.$$

Proof. Let $h \in \mathbb{R}$ be given. Define $g : \mathbb{R} \longrightarrow [0, \infty)$ by $g(x) := |f(x+h) - f(x)|$. For $i \in \mathbb{N}$, let $a_i := \sup\{i|h|, (i+1)|h|\}$. Then $\int_{i|h|}^{(i+1)|h|} g(x) \; dx \leq |h| \cdot g(a_i)$. Thus, for any $n \in \mathbb{N}$,

$$\begin{aligned}\int_{[-n|h|, n|h|]} g(x) \; dx = \sum_{i=-n}^{n-1} \int_{[i|h|, (i+1)|h|]} g(x) \; dx \leq |h| \sum_{i=-n}^{n-1} g(a_i) &= |h| \sum_{i=-n}^{n-1} |f(a_i + h) - f(a_i)| \\ &\leq |h| V(f, \mathcal{P}) \leq |h| C.\end{aligned}$$

Here $\mathcal{P}$ is a partition of $[-n|h|, n|h|]$ that includes the points $\{a_i, a_i + h\}_{i=-n}^{n-1}$. Because this is true for any $n \in \mathbb{N}$, we obtain the claim by the continuity of the Lebesgue integral. $\square$

5. Let $f \in L^\infty([0,1])$. Prove that

$$\lim_{p \longrightarrow \infty} ||f||_p = ||f||_\infty.$$

Proof. Because $||f||_p \leq ||f||_q$ for any $p \leq q$, the limit $L := \lim_{p \longrightarrow \infty} ||f||_p$ exists in $[0, \infty]$. But since $f \in L^\infty([0,1])$, $L \in [0, \infty)$. Notice that $||f||_p = \Big(\int_{[0,1]} |f|^p \Big)^{1/p} \leq \Big(||f||_\infty^p \Big)^{1/p} = ||f||_\infty$. So, $L \leq ||f||_\infty$. To show the reverse inequality: for $n \in \mathbb{N}$, define

$$E_n := \{x \in [0,1] : ||f||_\infty (1 - \frac{1}{n}) \leq |f(x)| \leq ||f||_\infty\}.$$

Then $m(E_n) > 0$. Thus

$$||f||_p^p = \int_{[0,1]} |f|^p = \int_{E_n} |f|^p + \int_{[0,1] \setminus E_n} |f|^p \geq ||f||_\infty^p \cdot \Big(1 - \frac{1}{n}\Big)^p \cdot m(E_n)$$

Hence $L = \lim_{p \longrightarrow \infty} ||f||_p \geq ||f||_\infty \Big(1 - \frac{1}{n}\Big)$. Since this is true for any $n \in \mathbb{N}$, we get $L \geq ||f||_\infty$ and therefore the claim. $\square$

6. Give an example of a function $f : [0,1] \longrightarrow \mathbb{R}$ such that $f \in L^p([0,1])$ for every $p \in [1, \infty)$, but $f \notin L^\infty([0,1])$.

Consider the function $f : [0,1] \longrightarrow \mathbb{R}$ defined by

$$f(x) := \begin{cases} \sum_{n=1}^{\infty} n\chi_{\left(2^{\frac{1}{n+1}}, 2^{\frac{1}{n}}\right]}(x) & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0. \end{cases}$$

Clearly, it is unbounded, and for any $p \geq 1$,

$$||f||_p^p = \sum_{n=1}^{\infty} \frac{n^p}{2^{n+1}} < \infty$$

by, for example, the root test. Hence $f \in L^p([0,1])$.

SUMMER 2020

Pavel Bleher

Problem 1. Let E be the set of all real numbers x on the interval $[0,1]$ such that in the decimal form of $x = 0.i_1i_2i_3\cdots$ the number 5 appears infinitely many times. Prove that $mE = 1$.

Proof. Let $F := \{x \in [0,1] : \text{the decimal expansion of } x \text{ has no } 5\}$. Then

$$F^c := [0,1]\backslash F = \frac{1}{10}(5,6) \sqcup \frac{1}{10^2}\bigsqcup_{\substack{i=0\\ i\neq 5}}^{9}(i5,i6) \sqcup \frac{1}{10^3}\bigsqcup_{\substack{i,j=0\\ i,j\neq 5}}^{9}(ij5,ij6) \sqcup \cdots$$

(Here we write 5 as $4.\bar{9}$ to make F compact)

Thus $m(F^c) = \frac{1}{9}\sum_{i=1}^{\infty}(\frac{9}{10})^i = 1$. This says $m(F) = 0$. For $n \in \mathbb{N}\cup\{0\}$, define

$$F_n := \{x \in [0,1] : \text{the decimal expansion of } x \text{ has no } 5 \text{ after the } n^{th} \text{ place}\}.$$

Then $F_n \subset \frac{1}{10^n}\bigcup_{i=0}^{10^n-1}(i+F)$. Because the Lebesgue measure is invariant under translation, we then get $m(F_n) = 0$ and therefore $m(\bigcup_{n=1}^{\infty}F_n) = 0$. Since $E = [0,1]\backslash(\bigcup_{n=1}^{\infty}F_n)$, we have the claim. □

Problem 2. Prove that if functions f_1, f_2 are absolutely continuous on $[0,1]$, then the function $f(x) = \max\{e^{f_1(x)}, |f_2(x)|\}$ is absolutely continuous on $[0,1]$ as well.

Proof. Notice that e^x is Lipschitz on $[0,1]$ (with a Lipschitz constant, for example, e. To see: the line $y = x + e$ lies above the curve $y = e^x$ over $[0,1]$. Or, the line $y = ex$ lies above the curve $y = e^x - 1$ over $[0,1]$.). Hence $e^{f_1(x)} \in AC[0,1]$. By the reverse triangle inequality, $|f_2| \in AC[0,1]$. This, together with the fact $|f(a) - f(b)| \leq |e^{f_1(b)} - e^{f_1(a)}| + ||f_2(b)| - |f_2(a)||$ for any $a, b \in [0,1]$, gives us $f \in AC[0,1]$. □

Problem 3. Prove that for any integrable function f on the interval $[a,b]$,

$$\lim_{n\longrightarrow\infty}\int_a^b f(x)\sin(\sin(nx))\ dx = 0.$$

Proof. Because the collection of all step functions on $[a,b]$ with rational values is dense (under $||\cdot||_1$ norm) in $L^1([a,b])$, and due to the linearity of the Lebesgue integral, it is enough to show

$$\lim_{n\longrightarrow\infty}\int_p^q \sin(\sin(nx))\ dx = 0 \text{ for any interval } [p,q]\subseteq[a,b].$$

By changing the variable, we get

$$\lim_{n\longrightarrow\infty}\left|\int_p^q \sin(\sin(nx))\ dx\right| = \lim_{n\longrightarrow\infty}\frac{1}{n}\left|\int_{np}^{nq}\sin(\sin(x))\ dx\right| \leq \lim_{n\longrightarrow\infty}\frac{2}{n} = 0.$$

Note: Over any interval, the area under the curve $y=\sin x$ is between -2 and 2. □

Problem 4. Prove that for any $1\leq p<\infty$ there is a constant $C_p>0$ such that for any integrable function f on $[0,1]$,

$$||\ln(1+|f|)||_p \leq C_p(1+||f||_1).$$

Proof. For a fixed $p\geq 1$, define the function $\varphi_p : [1,\infty)\longrightarrow\infty$ by $\varphi_p(x) := (p+\ln x)^p$. Since $\varphi_p''(x) = -\frac{p}{x^2}(p+\ln x)^{p-2}(1+\ln x) < 0$ on $[1,\infty)$, φ_p is concave. Let $f\in L^1([0,1])$. Since $1+|f|\geq 1$, by Jensen's inequality for concave functions, we get

$$\int_{[0,1]}\varphi_p\circ(1+|f|) \leq \varphi_p\Big(\int_{[0,1]}1+|f|\Big).$$

Because

$$||\ln(1+|f|)||_p^p \leq \int_{[0,1]}(p+\ln(1+|f|))^p = \int_{[0,1]}\varphi_p\circ(1+|f|)$$

and

$$\begin{aligned}\varphi_p\Big(\int_{[0,1]}1+|f|\Big) = \Big(p+\ln\big(\int_{[0,1]}1+|f|\big)\Big)^p &= \Big(p+\ln(1+||f||_1)\Big)^p \\ &= \Big(\ln e^p(1+||f||_1)\Big)^p \\ &\leq [e^p(1+||f||_1)]^p,\end{aligned}$$

we get the claim by letting $C_p = e^p$. □

Problem 5. Let $f \in L^2[0,1]$ and $m = \int_{[0,1]} f$. Prove that

$$m^2 + \Big(\int_0^1 |f(x) - m| dx\Big)^2 \le \int_0^1 f^2(x) dx.$$

Proof. Because $f \in L^2[0,1]$, $f \in L^1[0,1]$. Hence $|m| < \infty$ and

$$m^2 + \Big(\int_0^1 |f(x) - m| dx\Big)^2 \le m^2 + \int_0^1 (f - m)^2 = m^2 + \int_0^1 f^2 - 2m^2 + m^2 = \int_0^1 f^2.$$

The inequality is due to Jensen. $\square$

Problem 6. Let $f \in L^1(\mathbb{R})$ be such that

$$\int_a^{\frac{a+b}{2}} f = \int_{\frac{a+b}{2}}^b f, \forall \; -\infty < a < b < \infty.$$

Prove that then $f(x) = 0$ almost everywhere on $(-\infty, \infty)$.

Proof. We can assume $f \ge 0$. Suppose the set $E := f^{-1}((0,\infty))$ has a positive measure. Then there is $a > 0$ such that $M := \int_{-a}^a f > 0$. Then, by the hypothesis $\int_{-a}^{3a} f = 2M$ and therefore by induction, for any $n \in \mathbb{N}$, $\int_{-a}^{(2n-1)a} f = nM$. Since $M > 0$, $\int_{\mathbb{R}} f = \infty$. This contradicts the fact that f is integrable. Hence $m(E) = 0$. The general case follows by considering f_+ and f_- separately and $|f| = f_+ + f_-$. $\square$

WINTER 2020

Pavel Bleher

Problem 1. Let E be the set of real numbers x on the interval $[0,1]$ such that in the decimal expansion form of $x = 0.i_1 i_2 i_3 \cdots$, there is no 5, so that $i_k \neq 5$ for all k. Prove that

1. the set E is uncountable;
2. $m(E) = 0$, where $m(E)$ is the Lebesgue measure of E.

Proof. 1. The set $\{x \in [0,1] :$ the decimal expansion of x consists with only 0 and 1$\}$ is uncountable (by Cantor's diagonal argument) and is a subset of E.

2. Notice that

$$[0,1] \backslash E = \frac{1}{10}[5,6) \sqcup \frac{1}{10^2} \bigsqcup_{\substack{i=0 \\ i \neq 5}}^{9} [i5, i6) \sqcup \frac{1}{10^3} \bigsqcup_{\substack{i,j=0 \\ i,j \neq 5}}^{9} [ij5, ij6) \sqcup \frac{1}{10^4} \bigsqcup_{\substack{i,j,k=0 \\ i,j,k \neq 5}}^{9} [ijk5, ijk6) \sqcup \cdots$$

Thus $m([0,1] \backslash E) = \frac{1}{9} \sum_{i=1}^{\infty} (\frac{9}{10})^i = 1$ and therefore $m(E) = 0$.

□

Problem 2. Let E be the same set as in Problem 1. Calculate the fractal dimension of E, defined as

$$d := \lim_{k \to \infty} \frac{\ln N(k)}{\ln k},$$

where $N(k)$ is the smallest number of intervals of length $\frac{1}{k}$ covering E. Prove rigorously the existence of the limit.

Answer. Since

$$[0,1] \backslash E = \frac{1}{10}[5,6) \sqcup \frac{1}{10^2} \bigsqcup_{\substack{i=0 \\ i \neq 5}}^{9} [i5, i6) \sqcup \frac{1}{10^3} \bigsqcup_{\substack{i,j=0 \\ i,j \neq 5}}^{9} [ij5, ij6) \sqcup \frac{1}{10^4} \bigsqcup_{\substack{i,j,k=0 \\ i,j,k \neq 5}}^{9} [ijk5, ijk6) \sqcup \cdots,$$

for each $n \in \mathbb{N}$, E can be covered by 9^n number of intervals of length $1/10^n$. For $k \in \mathbb{N} \cup \{0\}$, let $n_k \in \mathbb{N}$ be such that $10^{n_k} \leq k < 10^{n_k+1}$. Then $9^{n_k} = N(10^{n_k}) \leq N(k) \leq N(10^{n_{k+1}}) = 9^{n_k+1}$. Hence

$$\frac{n_k \ln 9}{(n_k+1)\ln 10} \leq \frac{\ln N(k)}{\ln k} \leq \frac{(n_k+1)\ln 9}{n_k \ln 10}$$

and therefore by the squeeze theorem, $d = \frac{\ln 9}{\ln 10}$.

Problem 3. Let us enumerate all rational numbers on $[0,1]$, so that $\mathbb{Q} \cap [0,1] = \{r_1, r_2, r_3, \cdots\}$. Prove that for any $\alpha > -1$, the series

$$f(x) = \sum_{k=1}^{\infty} \frac{|x-r_k|^\alpha}{k^2}$$

is convergent for almost all x on $[0,1]$.

Proof. For $n \in \mathbb{N}$, define $f_n : [0,1] \longrightarrow [0,\infty)$ by $f_n(x) := \sum_{k=1}^{n} \frac{|x-r_k|^\alpha}{k^2}$. Then

$$\begin{aligned}
\int_{[0,1]} f_n(x)dx = \int_{[0,1]} \Big(\sum_{k=1}^{n} \frac{|x-r_k|^\alpha}{k^2}\Big)dx &= \sum_{k=1}^{n} \int_{[0,1]} \frac{|x-r_k|^\alpha}{k^2}dx \\
&= \sum_{k=1}^{n} \int_{[0,r_k]} \frac{(r_k-x)^\alpha}{k^2}dx + \int_{[r_k,1]} \frac{(x-r_k)^\alpha}{k^2}dx \\
&= \sum_{k=1}^{n} \frac{1}{k^2}\Big(\frac{r_k^{\alpha+1}}{\alpha+1} + \frac{(1+r_k)^{\alpha+1}}{\alpha+1}\Big) \\
&\leq \frac{1+2^{\alpha+1}}{\alpha+1}\sum_{k=1}^{\infty}\frac{1}{k^2} =: M < \infty. \qquad (7)
\end{aligned}$$

Since M does not depend on n, it is a uniform bound of the sequence $(\int_{[0,1]} f_n)_{n=1}^{\infty}$. Because $(f_n)_{n=1}^{\infty}$ is an increasing sequence, the pointwise limit $f(x) = \lim_{n\longrightarrow\infty} f_n(x)$ exists in $\overline{\mathbb{R}}$. Then by the Monotone Convergence Theorem and (0.1), it exists almost everywhere in $\mathbb{R}$. In other words, the given series converges for almost every $x \in [0,1]$. $\square$

Problem 4. Let $f(x)$ be the function defined in Problem 3. Prove that for any $\alpha > 0$, $f(x)$ is absolutely continuous on $[0,1]$.

Proof. Let f_n be as in Problem 3. Notice that each f_n is differentiable almost everywhere on $[0,1]$ (in fact, it is differentiable at all but finitely many points: $r_1, \cdots, r_n$) and $f_n'(x) = \sum_{k=1}^{n} \delta(r_k)\frac{\alpha(x-r_k)^{\alpha-1}}{k^2}$. Here $\delta(r_k) = 1$ if $x > r_k$ and $\delta(r_k) = -1$ if $x < r_k$. Notice that $|f_n'(x)| \leq \alpha \sum_{k=1}^{n} \frac{|x-r_k|^{\alpha-1}}{k^2}$. Because $\alpha > 0$, $\alpha - 1 > -1$. So, by replacing α with $\alpha - 1$ in Problem 3 we see that the series $g(x) := \sum_{k=1}^{\infty} \delta(r_k)\frac{\alpha(x-r_k)^{\alpha-1}}{k^2} (= \lim_{n\longrightarrow\infty} f_n'(x))$ converges absolutely for almost every $x \in [0,1]$. Hence

$$\begin{aligned} f(x) = \lim_{n\longrightarrow\infty} f_n(x) &= \lim_{n\longrightarrow\infty} \Big(f_n(0) + \int_0^x f_n'(t)\ dt \Big) \\ &= \lim_{n\longrightarrow\infty} f_n(0) + \lim_{n\longrightarrow\infty} \int_0^x f_n'(t)\ dt \\ &= f(0) + \int_0^x \lim_{n\longrightarrow\infty} f_n'(t)\ dt, \text{ by general LDCT} \\ &= f(0) + \int_0^x g(t)\ dt. \end{aligned}$$

By comparison test, $\lim_{n\longrightarrow 0} f_n(0) = f(0)$.

This says f is an indefinite integral over $[0,1]$. Hence $f \in AC([0,1])$. □

Problem 5. Prove that for any integrable function f on $(-\infty, \infty)$,

$$\lim_{n\longrightarrow\infty} \int_{\mathbb{R}} \cos(nx^2)\ f(x)\ dx = 0.$$

Proof. By the definition of Lebesgue integral of a non-negative function (we can assume $f \geq 0$. The general case follows by considering the positive and negative parts of f separately), if we can show the claim for any non-negative bounded function of finite support, then we are done. But any such a function can be approximated by (in this case by $||\cdot||_1$) step functions. Therefore by the linearity of the Lebesgue integral, if we can show the claim for the case $f(x) = 1$ then we are done. In fact

$$\begin{aligned} \lim_{n\longrightarrow\infty} \Big| \int_{\mathbb{R}} \cos(nx^2)\ dx \Big| &= \lim_{n\longrightarrow\infty} \Big| \int_{\mathbb{R}} \frac{\cos(x^2)}{\sqrt{n}}\ dx \Big|, \text{by letting } x = \frac{x}{\sqrt{n}}. \\ &= \lim_{n\longrightarrow\infty} \frac{1}{\sqrt{n}} \Big| \int_{\mathbb{R}} \cos(x^2)\ dx \Big| \\ &\leq \lim_{n\longrightarrow\infty} \frac{2}{\sqrt{n}} = 0. \end{aligned}$$

This implies $\lim_{n\longrightarrow\infty} \int_{\mathbb{R}} \cos(nx^2)\ dx = 0$, and therefore the claim. □

Problem 6. Let $f(x)$ be an integrable function on $(-\infty, \infty)$ and

$$f_n(x) = n \int_x^{x+\frac{1}{n}} f(t)\ dt,\ n = 1, 2, 3, \cdots$$

Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x) - f(x)| dx = 0.$$

Proof. • **Proof 1** Let $n \in \mathbf{N}$. Notice that

$$\begin{aligned}\int_{\mathbb{R}} |f_n(x)|\ dx = \sum_{k=-\infty}^{\infty} \int_0^{\frac{1}{n}} |f_n(x+\frac{k}{n})|\ dx &= \int_0^{\frac{1}{n}} \Big(\sum_{k=-\infty}^{\infty} |f_n(x+\frac{k}{n})| \Big)\ dx \\ &\leq \int_0^{\frac{1}{n}} \Big(\sum_{k=-\infty}^{\infty} n \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} |f(t))|\ dt \Big)\ dx \\ &= \int_0^{\frac{1}{n}} n \int_{\mathbb{R}} |f|\ dx = \int_{\mathbb{R}} |f|\ dx.\end{aligned}$$

Thus

$$||f_n||_1 \leq ||f||_1, \text{ for all } n \in \mathbb{N}. \tag{8}$$

Let $s(x) := \sum_{i=1}^{k} a_i \chi_{I_1}(x)$ be a step function. Then

$$s_n(x) = n \int_x^{x+\frac{1}{n}} \sum_{i=1}^{k} a_i \chi_{I_i}(t)\ dt = n \sum_{i=1}^{k} a_i m(I_i \cap [x, x+\frac{1}{n}]).$$

Let x be a point in the interior of $\cup_{i=1}^{k} I_i$. WLOG assume $x \in int(I_j) = (p_j, q_j)$ for some $j \in \{1, \cdots, k\}$ and $p_j, q_j \in \mathbb{R}$. Then there is $N \in \mathbb{N}$ (for example, take N such that $\frac{1}{N} \leq \min\{x - p_j, q_j - x\}$) such that for each $n > N$, $[x, x+\frac{1}{n}] \in I_j$. Hence for each $n > N$, $s_n(x) = a_j = s(x)$. If $x \notin \bigcup_{i=1}^{k} I_j$, then clearly $s(x) = s_n(x) = 0$ for any $n \in \mathbb{N}$. And the set of end-points of the intervals I_i's is finite. This says $s_n \xrightarrow{a.e} s$. That is $(s_n - s) \xrightarrow{a.e} 0$. Then by the Lebesgue Dominated Convergence Theorem, we have the claim for any step function:

$$\lim_{n\to\infty} ||s_n - s||_1 = 0. \tag{9}$$

Let $f \in L^1(\mathbb{R})$. Let $\epsilon > 0$ be given. Then we can find $s \in step(\mathbb{R})$ such that $||f - s||_1 < \frac{\epsilon}{2}$. Then by (0.2) we have $||f_n - s_n||_1 = ||(f_n - s_n)||_1 \leq ||f - s||_1 < \frac{\epsilon}{2}$. Thus

$$||f_n - f||_1 \leq ||f_n - s_n||_1 + ||s_n - s||_1 + ||s - f||_1 < ||s_n - s|| + \epsilon.$$

Then by taking the limit as $n \to 0$, we get $\lim_{n\to\infty} ||f_n - f||_1 = 0$ as $\epsilon > 0$ is arbitrary.

• **Proof 2** Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) := \int_0^x f(t)dt$. Because $f \in L^1(-\infty, \infty)$, this is well-defined and absolutely continuous. Thus $F'(x) = f(x)$ for almost every $x \in (-\infty, \infty)$. Notice that

$$\frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}} = f_n(x), \text{ for } n \in \mathbb{N}.$$

Thus $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in \mathbb{R}$. By Minkowski's inequality and (0.2) we have $||f_n - f||_1 \leq ||f_n||_1 + ||f||_1 \leq 2||f||_1 < \infty$. Therefore, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x) - f(x)| dx = 0.$$

$\square$

SUMMER 2019

Pavel Bleher

Problem 1. Let $E \subset \mathbb{R}$, $mE = 0$, and $P(x)$ a real polynomial of degree $d \geq 1$. Let

$$S = P^{-1}(E) = \{x \in \mathbb{R} : P(x) \in E\}.$$

Prove that $mS = 0$.

Proof. Let $I \subset \mathbb{R}$ be a bounded interval. Then $P^{-1}(I)$ consists of at-most d disjoint bounded intervals. Notice that if $l(I)$ goes to 0 (that is when I contracts to a point), $m(P^{-1}(I))$ goes to 0 as well. Otherwise, there will be a point a such that $P^{-1}(\{a\})$ has an interval. But that contradicts $d \geq 1$. Thus, for a given $\epsilon > 0$, there is $d(\epsilon) > 0$ such that if $l(I) < d(\epsilon)$, then $m(P^{-1}(I)) < \epsilon$ (basically because P is continuous and P' is bounded on compact sets). Because $m(E) = 0$, we can find a countable collection $\{I_{i,\epsilon}\}_{i \in \mathbb{N}}$ of disjoint open intervals that form an open cover of E such that $l(I_{i,\epsilon}) \leq d(\frac{\epsilon}{2^i})$. Then $S = P^{-1}(E) \subset P^{-1}(\bigsqcup_{i=1}^{\infty} I_{i,\epsilon}) = \bigsqcup_{i=1}^{\infty} P^{-1}(I_{i,\epsilon})$ and therefore $m(S) \leq \sum_{i=1}^{\infty} m(P^{-1}(I_{i,\epsilon})) \leq \epsilon$ (S is measurable as E and f are measurable). Because this is true for any $\epsilon > 0$, $m(S) = 0$. $\square$

Problem 2. Prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{|\tan(nx)|^{1/2}}{n^2}$$

is finite almost everywhere on the interval $[0, \pi]$.

Proof. If we can prove $f \in L^1([0, \pi])$, then we are done. For $k \in \mathbb{N}$, define

$$f_k(x) := \sum_{n=1}^{k} \frac{|\tan(nx)|^{1/2}}{n^2}.$$

Then

$$\begin{aligned}
\int_0^{\pi} f_k(x)\ dx &= \sum_{n=1}^{k} \int_0^{\pi} \frac{|\tan(nx)|^{1/2}}{n^2}\ dx = \sum_{n=1}^{k} \int_0^{n\pi} \frac{|\tan(x)|^{1/2}}{n^3}\ dx \\
&\leq \sum_{n=1}^{k} \frac{1}{n^3} \sum_{i=0}^{n-1} \int_{i\pi}^{(i+1)\pi} \left|(i+\frac{1}{2})\pi - x\right|^{-\frac{1}{2}}\ dx \\
&\Big(\because\ |\tan(nx)|^{1/2} \leq \left|\frac{\pi}{2} - x\right|^{-\frac{1}{2}} \text{ on } [0,\pi]\Big) \\
&= \sum_{n=1}^{k} \frac{1}{n^3} \sum_{i=0}^{n-1} 2\sqrt{2\pi} = 2\sqrt{2\pi} \sum_{n=1}^{k} \frac{1}{n^2} \\
&\leq 2\sqrt{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} =: M < \infty.
\end{aligned}$$

Since this is true for any $k \in \mathbb{N}$, by the Monotone Convergence Theorem, $\int_0^{\pi} f(x) \leq M < \infty$.
$\square$

Problem 3. Prove that for any integrable function f on the line,

$$\lim_{n\to\infty} \int_{\infty}^{\infty} f(x)\ [\sin(nx)]^n\ dx = 0.$$

Proof. • **Proof 1.** By the density of *step*$(\mathbb{R})$ in $L^1(\mathbb{R})$ under $||\cdot||_1$ and the linearity of the Lebesgue integral, it is enough to show the claim for the case when $f \in$ *step*$(\mathbb{R})$. So let $f(x) \overset{\text{a.e}}{=} \chi_{[a,b]}(x)$ for any interval $[a,b]$. By the change of variables, we get

$$\int_{-\infty}^{\infty} \chi_{[a,b]}(x)\cdot[\sin(nx)]^n\ dx = 1/n \int_{na}^{nb} (\sin x)^n\ dx.$$

Let $\alpha_n = 1 - 1/\ln(n+1)$. Then $\alpha_n \nearrow 1$ and $\lim_{n\to\infty} \alpha_n^n = 0$. For $n \in \mathbb{N}$, choose $E_n \subset [na, nb]$ so that $\alpha_n \geq \sup\{|\sin x| : x \in [na,nb]\backslash E_n\}$ and $\int_{E_n} |\sin x|^n\ dx \leq a_n \leq 1$.

(Idea: Depending on $n \in \mathbb{N}$, choose a finite collection of disjoint intervals around the points $[na,nb] \cap \sin^{-1}(\{\pm 1\})$ with the above given properties).

Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \int_{na}^{nb} |\sin x|^n\ dx &= \lim_{n\to\infty} \Big(\frac{1}{n} \int_{(na,nb)\backslash E_n} |\sin x|^n\ dx + \frac{1}{n} \int_{E_n} |\sin x|^n\ dx\Big) \\
&\leq \lim_{n\to\infty} \Big(\frac{1}{n}\cdot \alpha_n^n \cdot (nb - na) + \frac{1}{n}\cdot 1\Big) \\
&= \lim_{n\to\infty} \alpha_n^n (b-a) \\
&= 0.
\end{aligned}$$

• **Proof 2.** Instead of creating the sequence $(E_n)_{n=1}$ of sets as above, use the fact $m\Big(\bigcup_{n=1}^{\infty} \sin^{-1}(\{\pm 1\})/n\Big) =$ 0 and the Lebesgue Dominated Convergence Theorem on
$f_n(x) := \chi_{[a,b]}(x) \cdot \sin^n(nx)$. □

Problem 4. Let $f \geq 0$ be a measurable function on $E \subset \mathbb{R}$ and $mE < \infty$. Let

$$E_n = \{x \in E : f(x) \geq n\}.$$

Prove that f is integrable if and only if

$$\sum_{n=1}^{\infty} mE_n < \infty.$$

Proof. Notice that $\sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} n \cdot m(E_n \backslash E_{n+1})$ and

$$\sum_{n=1}^{\infty} n \cdot m(E_n \backslash E_{n+1}) \leq \sum_{n=1}^{\infty} \int_{E_n \backslash E_{n+1}} f \leq \sum_{n=1}^{\infty} (n+1) \cdot m(E_n \backslash E_{n+1}). \tag{10}$$

Because $m(E) < \infty$,

$$f \text{ is integrable iff } \int_{E_1} f < \infty \text{ iff } \sum_{n=1}^{\infty} \int_{E_n \backslash E_{n+1}} f < \infty.$$

Then by the comparison tests (standard and limit comparison tests) and (0.4), we get the claim. □

Problem 5. Let f be a measurable function on $[0, 1]$ and

$$A = \{x \in [0, 1] : f(x) \in \mathbb{Z}\}.$$

Prove that the set A is measurable and

$$\lim_{n\to\infty} \int_0^1 |\cos(\pi f(x))|^n \, dx = mA.$$

Proof. Because $A = \bigsqcup_{n\in\mathbb{Z}} f^{-1}(\{n\})$, it is measurable as a countable union of measurable sets is measurable.

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 |\cos(\pi f(x))|^n \, dx &= \lim_{n\to\infty} \int_{[0,1]\backslash A} |\cos(\pi f(x))|^n \, dx + \lim_{n\to\infty} \int_A |\cos(\pi f(x))|^n \, dx \\
&= 0 + \lim_{n\to\infty} \int_A 1 \, dx, \ \because |\cos(\pi f(x)\chi_{[0,1]\backslash A}(x))|^n \to 0 \text{ and by LDCT.} \\
&= m(A).
\end{aligned}$$

□

Problem 6. Let $f_n(x) = \cos(nx)$ on $[0, 2\pi]$. Prove that there is no sub-sequence f_{n_k} converging almost everywhere in $[0, 2\pi]$.

Proof. Let $(f_{n_k})_{k\in\mathbb{N}}$ be a sub-sequence of $(f_n)_{n\in\mathbb{N}}$. For the sake of contradiction, let's assume that it converges almost everywhere on $[0, 2\pi]$. Then the sequence $\big((f_{n_k} - f_{n_{k+1}}\big)^2)_{k=0}^{\infty}$ converges to 0 almost everywhere on $[0, 2\pi]$. Therefore, by LDCT,

$$0 = \lim_{k\to\infty} \int_0^{2\pi} (f_{n_k} - f_{n_{k+1}})^2 = \lim_{k\to\infty} \int_0^{2\pi} \Big(\cos(n_k x) - \cos(n_{k+1} x) \Big)^2 \, dx = 2\pi.$$

Hence, the contradiction. □

WINTER 2019

Pavel Bleher

Problem 1. Let F be a bounded, closed set on the line, and

$$F_\epsilon = \bigcup_{x \in F} [x-\epsilon, x+\epsilon], \ \epsilon > 0.$$

Prove that $\lim_{\epsilon \to 0} mF_\epsilon = mF$, where m is the Lebesgue measure.

Proof. Because $F = \bigcap_{k=1} F_{1/k}$, we have the claim by the continuity of Lebesgue measure (and, if you want, the squeeze theorem). □

Problem 2. Let $E_1 \subset E_2 \subset \cdots$ be an increasing sequence of Lebesgue measurable sets on the line, such that the set $E = \bigcup_{n=1}^\infty E_n$ has a finite Lebesgue measure, $mE < \infty$. Prove that for any set $A \subset \mathbb{R}$ (not necessarily measurable),

$$\lim_{n \to \infty} m^*(A \cap E_n) = m^*(A \cap E).$$

Proof. Because $E_n \cap A \subset E \cap A$ for all $n \in \mathbb{N}$, we have

$$\limsup_{n \to \infty} m^*(A \cap E_n) \leq m^*(A \cap E).$$

On the other hand, because $m^*(A \cap E) < m(E \backslash E_n) + m^*(A \cap E_n)$, we have

$$m^*(A \cap E) \leq \liminf_{n \to \infty} m^*(A \cap E_n).$$

Thus

$$\lim_{n \to \infty} m^*(A \cap E_n) = m^*(A \cap E).$$

□

Problem 3. Let $f \in L^1(-\infty,\infty)$. Find the limit,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\Big(\cos x+\frac{\sin^2 x}{2}\Big)^n\,dx,$$

and justify your answer.

Answer. Notice that $0 \le (1-\cos x)^2$ implies $\cos x+\sin^2 x/2 \le 1$. Hence, the maximum value the function $g(x) := |\cos x+\sin^2 x/2|$ can obtain is 1, and $g^{-1}(\{1\})$ is countable. Because $f \in L^1(-\infty,\infty)$, $f^{-1}(\{\infty\})$ has measure zero. Therefore, the set $E := g^{-1}(\{1\}) \cup f^{-1}(\{\infty\})$ also has measure zero. Define the sequence $(f_n)_{n=1}$ of measurable functions on $\mathbb{R}$ by $f_n(x) := f(x)\Big(\cos x+\frac{\sin^2 x}{2}\Big)^n$. Then it is integrable as $|f_n| \le |f|$, and $f_n \to 0$ for all $x \in \mathbb{R}\backslash E$. Thus, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\Big(\cos x+\frac{\sin^2 x}{2}\Big)^n\,dx = 0.$$

Problem 4. Prove that for any integrable function f on the interval $[a,b]$,

$$\lim_{n\to\infty}\int_a^b \frac{\cos(nx)}{1+\sin^2(nx)} f(x)\,dx = 0.$$

Proof. Due to the separability of $L^1[a,b]$ by $\mathcal{S}'[a,b]$ (the collection of all step functions on $[a,b]$ with rational values) under $L^1[a,b]$ norm, and the linearity of the Lebesgue integral, if we can show the claim when $f(x)=1$ on any sub-interval $[p,q] \subseteq [a,b]$, then we are done. In fact

$$\lim_{n\to\infty}\int_p^q \frac{\cos(nx)}{1+\sin^2(nx)}\,dx = \lim_{n\to\infty}\Big[\frac{\tan^{-1}(\sin(nx))}{n}\Big]_p^q \le \lim_{n\to\infty}\frac{\pi}{n} = 0.$$

□

Problem 5. Let $E \subset [0,1]$ be a measurable set such that there exists $\epsilon > 0$ such that

$$m(E\cap[a,b]) \ge \epsilon|b-a|$$

for all $[a,b] \subset [0,1]$. Prove that $mE = 1$.

Proof. We show $m(E^c) = m([0,1]\backslash E) = 0$. For $k > 1$, choose collection $\{I_n\}_{n=1}$ of open disjoint intervals such that $E^c \subset \bigsqcup_{n=1} I_n$ and $\sum_{n=1} m(I_n) < m(E^c)+\epsilon/k$. Then

$$\epsilon\sum_{n=1} m(I_n) \le \sum_{n=1} m(E\cap I_n) < m\big(\big(\bigsqcup_{n=1} I_n\big)\backslash E^c\big) = \sum_{n=1} m(I_n) - m(E^c) < \frac{\epsilon}{k}.$$

This implies $\sum_{n=1} m(I_n) \leq 1/k$ for any $k > 1$. Thus $\sum_{n=1} m(I_n) = 0$. Because $E^c \subset \bigsqcup_{n=1} I_n$, $m(E^c) = 0$. □

Problem 6. Prove that for any function $f \in L^2[0,1]$,

$$|| \ln(1 + |f|)||_{L^1[0,1]} \leq ||f||_{L^2[0,1]}.$$

Proof. By Jensen's inequality (on $\varphi(x) = x^2$),

$$|| \ln(1 + |f|)||^2_{L^1[0,1]} \leq \int \big(\ln(1 + |f|)\big)^2. \tag{11}$$

A fact:

$$\text{For any } x \geq 0,\ \ln(1 + x) \leq x. \tag{12}$$

By combining (11) and (12), we obtain the claim. □

SUMMER 2018

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Problem 1. Let E be the set of real numbers x on the interval $[0,1]$ such that in the decimal form of $x = 0.i_1 i_2 i_3 \cdots$ there is no string of four consecutive digits 2018. Prove that

1. the set E is uncountable
2. $mE = 0$, where mE is the Lebesgue measure of E.

Proof. 1. The set $\{x \in [0,1] :$ the decimal expansion of x consists with only 3 and 4$\}$ is uncountable (by Cantor's diagonal argument), and is a subset of E.

2. Notice that

$$[0,1]\backslash E = \frac{1}{10^4}[2018, 2019) \sqcup \frac{1}{10^8} \bigsqcup_{\substack{i=0 \\ i \neq 2018}}^{10^4-1} [i2018, i2019) \sqcup \frac{1}{10^{12}} \bigsqcup_{\substack{i,j=0 \\ i,j \neq 2018}}^{10^4-1} [ij2018, ij2019) \sqcup \cdots$$

Thus $m([0,1]\backslash E) = \frac{1}{10^4-1} \sum_{i=1}^{\infty} (\frac{10^4-1}{10^4})^i = 1$, and therefore $m(E) = 0$.

$\square$

Problem 2. Prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{|x - n^{-1}|^{1/2}}{n^2}$$

is absolutely continuous on $[0,1]$.

Proof. Because $|x - 1/n| \le 1$ for any $x \in [0,1]$, the series $\sum_{n=1}^{\infty} \frac{|x-n^{-1}|^{1/2}}{n^2}$ converges absolutely for all $x \in [0,1]$. Thus for any $a, b \in [0,1]$, $a < b$ we have

$$|f(b) - f(a)| = \Big| \sum_{n=1}^{\infty} \frac{|b - n^{-1}|^{1/2}}{n^2} - \sum_{n=1}^{\infty} \frac{|a - n^{-1}|^{1/2}}{n^2} \Big| \le \sum_{n=1}^{\infty} \frac{b-a}{2n^{3/2}} \le M(b-a).$$

Here $M = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. Thus f is Lipschitz and therefore absolutely continuous. □

Problem 3. Prove that for any integrable function f on the interval $[a, b]$,

$$\lim_{n\to\infty} \int_a^b f(x)\sin^4(nx)dx = \frac{3}{8}\int_a^b f(x)\ dx.$$

Proof. We have that

$$\sin^4(nx) = \frac{3 - 4\cos(2nx) + \cos(4nx)}{8}.$$

Thus, if we can show that

$$\lim_{n\to\infty} \int_a^b f(x)\cos(knx)dx = 0,$$

for any $f \in L^1[a, b]$ and any constant $k \in \mathbb{R}$, then we are done. Because $\mathcal{S}[a, b]$ - the collection of all step functions on $[a, b]$, is dense in $L^1[a, b]$ under $L^1[a, b]$ norm, and the Lebesgue integral is linear, if we can show that

$$\lim_{n\to\infty} \int_p^q \cos(nx)dx = 0 \text{ for any } p, q \in \mathbb{R}, p < q,$$

then we are done. In fact

$$\lim_{n\to\infty} \Big| \int_p^q \cos(nx)dx \Big| = \lim_{n\to\infty} \frac{1}{n} \Big| \int_{np}^{nq} \cos x dx \Big| \le \lim_{n\to\infty} \frac{2}{n} = 0.$$

□

Problem 4. Let $f \in L^2(-\infty, \infty)$. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)f(x+n)dx = 0.$$

Proof. Because $\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} f(x+n)dx$ for any $n \in \mathbb{N}$, from Holder's inequality, we have

$$\int_{\mathbb{R}} \big|f(x)f(x+n)\big|dx \le ||f(x)||_2 \cdot ||f(x+n)||_2 = ||f(x)||_2^2 < \infty.$$

Thus, $f(x)f(x+n) \in L^1(-\infty, \infty)$ for all $n \in \mathbb{N}$. Therefore, if we can show

$$\lim_{n\to\infty} \int_{-N}^{N} f(x)f(x+n)dx = 0 \text{ for any } N \in \mathbb{N},$$

, then we are done.

And if we can show this claim for any simple function on $[-N, N]$, then we are done by the simple approximation theorem. Let $\varphi(x) = \sum_{i=1}^k a_i \chi_{E_i}(x)$ where $E_i \subset [-N, N]$ for all $i = 1, \cdots k$. Then $\varphi(x+n) = \sum_{i=1}^k a_i \chi_{E_i - n}(x)$. Hence,

$$\int_{[-N,N]} \varphi(x)\varphi(x+n)dx = \sum_{i=1,j=i}^{k} a_i a_j m(E_i \cap (E_j - n)).$$

Because each E_j is bounded, for large enough n (say $n > 2N$) we have that $E_i \cap (E_j - n) = \emptyset$ for all $i, j \in \{1, \cdots k\}$. Thus,

$$\lim_{n\to\infty} \int_{[-N,N]} \varphi(x)\varphi(x+n)dx = 0.$$

$\square$

Problem 5. Let $f \in L^3[0, \pi]$ and

$$g(x) = \frac{f(x)}{|\sin x|^{0.1}}.$$

Prove that $||g||_2 \leq 2||f||_3$.

Proof. Let $\theta \in [0, \pi/2]$ be such that $\theta^3 = \sin^2 \theta$. Then simple calculations show that $\pi/4 < \theta < \pi/3$. So, $x^3 < \sin^2 x$ on $[0, \theta)$ and $\sin^2 x < x^3$ on $(\theta, \pi/2]$. Hence

$$\begin{aligned}
\int_0^\pi \frac{dx}{\sin^{0.6} x} &= 2\int_0^{\pi/2} \frac{dx}{\sin^{0.6} x} \quad \because\ y = 1/\sin^{0.6} x \text{ is symmetric around } x = \pi/2 \\
&\leq 2\Big(\int_0^\theta \frac{dx}{x^{0.9}} + \int_\theta^{\pi/2} \frac{dx}{\sin^{0.6} x}\Big) \\
&\leq 2\Big(\big[x^{0.1}\big]_0^\theta + \frac{\pi/2 - \theta}{\sin^{0.6}\theta}\Big) \leq 2\Big(\theta^{0.1} + \frac{\pi/4}{\theta^{0.9}}\Big) \leq 2\big((\pi/3)^{0.1} + (\pi/4)^{0.1}\big) < 2(2+2) \\
&< 8. \qquad (13)
\end{aligned}$$

This says that $\sin^{0.2} x \in L^3[0, \pi]$. Then by (generalized) Holder's inequality, we have

$$||g||_2 = \Big(\int_{[0,\pi]} \frac{f(x)^2}{\sin^{0.2}(x)} dx\Big)^{1/2} \leq ||f||_3 \Big(\int_{[0,\pi]} \frac{dx}{\sin^{0.6} x}\Big)^{1/6} \leq ||f||_3 8^{1/6} < 2||f||_3.$$

$\square$

Problem 6. Let $f(x)$ be an integrable function on $(-\infty, \infty)$ and $n \geq 1$ an integer. Define

$$f_n(x) = n\int_x^{x+\frac{1}{n}} f(t)dt.$$

Prove that $||f_n||_1 \leq ||f||_1$.

Proof. • **Proof 1**. By Tonelli,

$$||f||_n \leq \int_{\mathbb{R}} n \int_{\mathbb{R}} |f(t)| \cdot \chi_{[x,x+\frac{1}{n}]} dt dx = \int_{\mathbb{R}} n \int_{\mathbb{R}} |f(t)| \cdot \chi_{[x,x+\frac{1}{n}]} dx dt = \int_{\mathbb{R}} n |f(t)| \frac{1}{n} = ||f||_1.$$

• **Proof 2**.

$$\begin{aligned} ||f||_n = \int_{\mathbb{R}} |f_n(x)| \; dx &= \sum_{k=-\infty}^{\infty} \int_0^{\frac{1}{n}} |f_n(x + \frac{k}{n})| \; dx = \int_0^{\frac{1}{n}} \Big(\sum_{k=-\infty}^{\infty} \big| f_n(x + \frac{k}{n}) \big| \Big) \; dx \\ &\leq \int_0^{\frac{1}{n}} \Big(\sum_{k=-\infty}^{\infty} n \int_{x+\frac{k}{n}}^{x+\frac{k+1}{n}} |f(t)| \; dt \Big) \; dx \\ &= \int_0^{\frac{1}{n}} n ||f||_1 \; dx = ||f||_1. \end{aligned}$$

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